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The conducting sheet analogy for Laplace and Poisson equations

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THE CONDUCTING SHEET ANALOGY FOR LAPLACE AND POISSON EQUATIONS

BY

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A

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ABSTRACT

This investigation determined the accuracy to be expected with conducting sheet analogy solutions for Laplace and Poisson equations using simple and relatively inexpensive instrumentation. Several problems were solved experimentally and compared to known theoretical solutions to check variables in the technique and the accuracy of the experimental solution. It was found that if proper procedures are followed, solutions can be obtained rather rapidly and with an error of less than 5% which compares favorably with accuracy obtained by other investigators with more complex, precise equipment.

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The author is also deeply indebted to Dr. Howard L. Furr of the Civil Engineering Department who was instrumental in the selection of the investigation subject and in obtaining references and equipment.

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I. INTRODUCTION

The purpose of this thesis is to investigate inexpensive and reasonably accurate instrumentation which will give consistant results when solving problems with the conducting sheet analogy. A secondary purpose is to compile and outline the available information on the conducting sheet analogy.

The use of analogies in solving complex problems has long been a powerful tool available to the engineer. The conducting sheet analogy appears to be one of the most useful since it is applicable to so many problems in different fields of engineering. This analogy can be used to solve any two dimensional problem that can be expressed as a Laplace's or Poisson's equation since these equations are identical in form to the equations governing the flow of electrical current in a uniformly conducting medium. Some examples of phenomena satisfied by these equations are fluid flow through soils (seepage nets), which is very important to the civil engineer and also to the geophysicist for ground seepage problems, air flow patterns in aerodynamics, heat flow problems and the stress distribution of sections in torsion.

The uniformly conducting medium can be represented by Western Union "Teledeltos", an electrosensitive recording paper developed by Western Union (1). This paper has enabled the engineer to represent a uniformly conducting sheet with an economical paper that can be cut with a scissors to any desired shape. The introduction of "Teledeltos" paper has encouraged several investigators to solve engineering problems with the conducting sheet analogy.

Highly conducting silver paint, General Cement #21-1, developed for use in printed circuits, makes the establishment of necessary boundary conditions relatively simple. The boundary is painted on the conducting sheet to any desired boundary shape. Contact is made to the boundaries for input and ground leads by using a drop of solder to hold the wire to the painted boundary.

Most investigators have used relatively complex potentiometer circuits to measure potential. The use of an ordinary voltmeter would make the analogy especially rapid and useful for fast solutions of complex problems provided reasonable accuracy can be obtained. Fairly accurate voltmeter solutions could be very useful in finding starting values for more accurate solutions by refined numerical mathematical analysis.

II. REVIEW OF LITERATURE

1. THE CONDUCTING SHEET

Field distributions have been investigated by using a conducting sheet to which electrodes representing boundary conditions are attached. The first published work of this method was done by Kirchhoff (2) in 1845. Kirchhoff used a thin copper disc and plotted equipotential lines using two copper wires and a compass needle as a current balance indication. After Kirchhoff's work there is a period of about 100 years where very little was done with this rather simple technique. The development of a commercial conducting paper (1) for use in teleprinters, which is particularly suitable to field mapping has promoted the use of the conducting sheet method.

Previous to "Teledeltos" the copper sheet or colloidal graphite sprayed on cardboard were the primary methods of obtaining a conducting sheet. The variations in conductivity and the difficulty of making contact with the sheet, which were the principal disadvantages of the older methods are virtually non-existant in "Teledeltos" making it very satisfactory as a conducting sheet.

Information on "Teledeltos" paper was obtained from the Western Union Company (3). "Teledeltos" is a dry electrosensitive recording paper consisting of a black carbon filled conducting base coated with lacquer-bound pigment, which changes from light gray to black under the influence of electrical energy, and a thin aluminum backing. The useful part of the material is the black conducting base since the front and back coatings are relatively non-conducting and do not enter into the field-mapping process. The aluminum coating is quite thin and is more easily penetrated by probes or fixed electrical connections than the recording coating.

In field-mapping applications the required contact to the paper is made with silver conducting paint using acetone or methyl-ethyl ketone as the solvent, either of which penetrates the coating sufficiently to make good contact with the silver solids and the black base paper. The aluminum coating may be washed off where desired with any good lacquer solvent.

"Teledeltos" is available in 31" rolls in two grades, type "L" and type "H". The resistance of type "L" ranges between 1,500 and 4,000 ohms per square among samples with the average at about 2,000 ohms per square. Type "H" paper has a resistance about 10 times that of type "L" with the range about the same and an average value of about 20,000 ohms per square.

The resistance is measured by cutting a square of any size and measuring the resistance from face to face. The resistance in the transverse direction is slightly greater than the resistance in the longitudinal direction. This variation is due to the manufacturing process. The ratio of the transverse to the longitudinal resistance is about the same throughout a roll and once determined for a given roll can be used for the entire roll. This ratio is usually between 1.06 and 1.15.

The resistance of "Teledeltos"paper is almost insensitive to temperature changes, the temperature coefficient being negative and about 0.2% per degree centigrade. Any change in resistance noted when heating the paper is due to the change in the moisture content which has a much larger effect on the resistance than temperature changes. Exacting solutions should be carried out in controlled humidity conditions, and the placing of moist hands on the paper when probing should be avoided.

Mapping on "Teledeltos" should be done with relatively low voltage and moderate currents. The maximum dissipation desirable is about one third watt per square inch to reduce heating. Although the paper can take temperatures above 100°C without physical damage, the change of resistance with drying is a significant factor. Either ac or dc current may be used since the paper has no polarization effects.

"Teledeltos" as a field-mapping material has the advantage of being dry, safe to use because of low voltages, convenient due to low current values and no polarization limitations and economical since the paper and equipment are inexpensive.

2. SOLUTION AND TECHNIQUES INVOLVING EQUATIONS OF THE LAPLACE TYPE

Many problems of physics and engineering are described mathematically by an equation of the type $\nabla^2 u = 0$, where ∇^2 is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ or in two dimensions ∇^2 can be expressed by $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. "This equation describes the electrostatic field near electrodes, the magnetic field near magnets, the flow of current in a conducting solid or liquid, the mass flow of gases or liquids, or the flow of heat under steady state conditions". (4)

The fact that a conducting sheet will satisfy Laplace's equation will be shown (5). Referring to Figure 1, the differential element of a conducting sheet showing the current flow in the x and y direction, the continuity equation is applied:

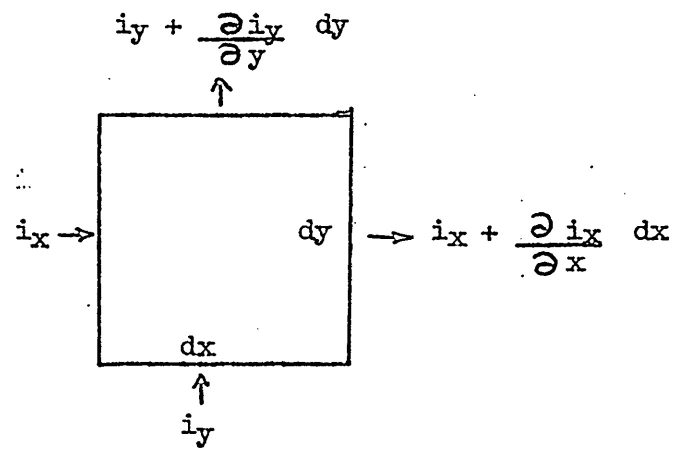


Figure 1

Differential Element of a Conducting Sheet

$$i_x + i_y - \left(i_x + \frac{\partial i_x}{\partial x} dx\right) - \left(i_y + \frac{\partial i_y}{\partial y} dy\right) = 0 \qquad (1)$$

simplifying:

$$\frac{\partial i_x}{\partial x} dx + \frac{\partial i_y}{\partial y} dy = 0 \qquad (2)$$

using Ohms law:

$$i_x = \frac{-1}{p_x} \frac{\partial v}{\partial x} dy; \; i_y = \frac{-1}{p_y} \frac{\partial v}{\partial y} dx \qquad (3)$$

where p_x and p_y are the resistances in the x and y directions respectively. By substituting 3 in equation 2 and simplifying:

$$\frac{p_y}{p_x} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (4)$$

If $P_x = P_y$ equation 4 simplifies to $\nabla^2 V = 0$; however, if P_y and P_x are not equal a transformation can be used to express the original equation, $\nabla^2 U$, in the same form as equation 4. Let $x_t = (P_y/P_x)^{1/2}\, x$ where x_t is the transformed value of X. When this is substituted into $\nabla^2 U$ the resulting equation is:

$$\frac{P_y}{P_x}\frac{\partial^2 U}{\partial x_t^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (5)$$

Equation 5 is analogous to equation 4 and if the x values of the original problem are transformed the problem can be worked with the conducting sheet analogy. The conducting sheet solution is then taken back to the true solution by taking the x_t values back to their original values of x.

This simple scale distortion enables the investigator to compensate for the anistropic resistance of "Teledeltos" paper. Once the ratio is determined it is a simple matter to alter the scale of a problem to increase the accuracy of the solution.

Since the conducting sheet satisfied Laplace's equation it is necessary only to impose boundary conditions electrically analogous to those of the actual problem to be solved to obtain a complete analogy between the conducting sheet and other physical problems of the Laplace type.

Liebmann (4) shows a solution for equipotential lines between accelerating electrodes of a high voltage discharge tube obtaining an accuracy of 2%. Furr (6) has solved two heat flow problems with fair accuracy. Recently Loeb (7) has used the conducting sheet analogy to solve for the characteristics of hydrostatic bearings and obtained results that were in error by less than 1%. Several authors (8) (9) have used the conducting sheet analogy for solving water flow and seepage problems and these solutions will be treated in some detail as they illustrate the technique of solution for any Laplace equation.

The basic equations for two dimensional fluid flow in an isotropic medium are $\nabla^2 h = 0$ and $\nabla^2 \psi = 0$, where h is the potential function and ψ is the stream flow function. These are both Laplace type equations and both can be solved with the electrical conducting sheet.

Peattie (8) has used "Teledeltos" paper to solve the flow net under a sheet pile dam with an underlying impervious stratum. A circuit diagram of his setup for the equipotential lines is shown below in Figure 2.

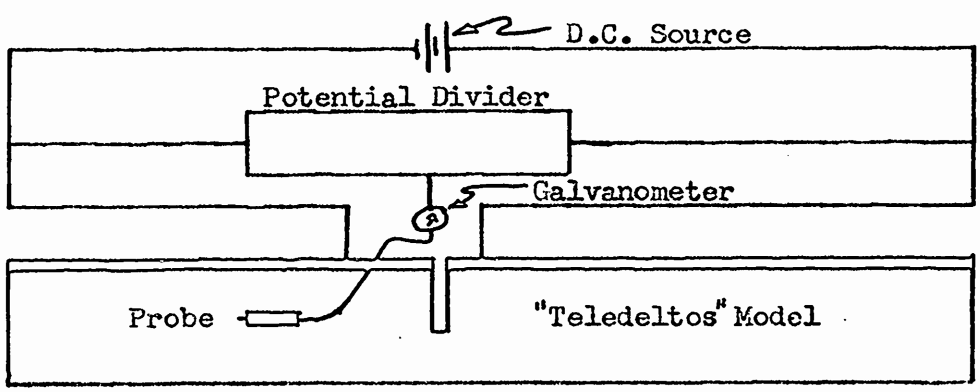


Figure 2

Peattie's Circuit for Sheet Pile Dam

The circuit was energized with a 2 volt battery and the galvanometer had a built-in sensitivity control. The probe used was a refillable drawing pencil with a wire soldered to the metallic body of the pencil and the graphite point was left in position giving a convenient method for marking the equipotential lines.

One of the biggest problems encountered was making contact with the paper on the boundaries. This was accomplished by cutting the model out with a quarter inch border at the input and ground boundaries and painting these borders with silver conducting paint. Small wires were stapled at intervals along the boundaries and a drop of silver paint was applied to each of the stapled connections.

When the model was laid out the horizontal scale was distorted by the square root of the ratio of the vertical resistance to the horizontal resistance. After the net was plotted the equipotentials were brought back to the natural scale giving the correct net for an isotropic soil.

The equipotential lines were traced out by setting the potential to the required value and moving the probe until a null was obtained. The equipotential lines were sketched in this manner. The flow lines were then sketched in graphically; however, they could have been plotted by using a second model in which the insulating and conducting boundaries are interchanged.

A comparison between the conducting paper solution and a relaxation solution showed close agreement. Peattie estimates that the conducting sheet for this type of problem should give results that are in error by only 2%.

The coefficient of permeability of soil is frequently greater in the horizontal direction (K_x) than in the vertical direction (K_y).

If all dimensions in the x direction are multiplied by $(K_y / K_x)^{1/2}$ and the resulting shape treated as an isotropic soil, the resulting net will be the correct one when replotted to the natural scale. This is obviously the same type of transformation that was made for the anistropic resistances in the conducting sheet analogy. Problems may then be solved in anistropic soils by applying a scale distortion. It must be remembered that there are two scale distortions necessary; one for the different resistances of the paper and one for the coefficients of permeability of the soil.

Becker (9) has done rather extensive work with the conducting sheet analogy for the solution of seepage problems in both isotropic and orthotropic soils. His apparatus was similar to that of Peattie, except that a 5 volt battery was used for the input source.

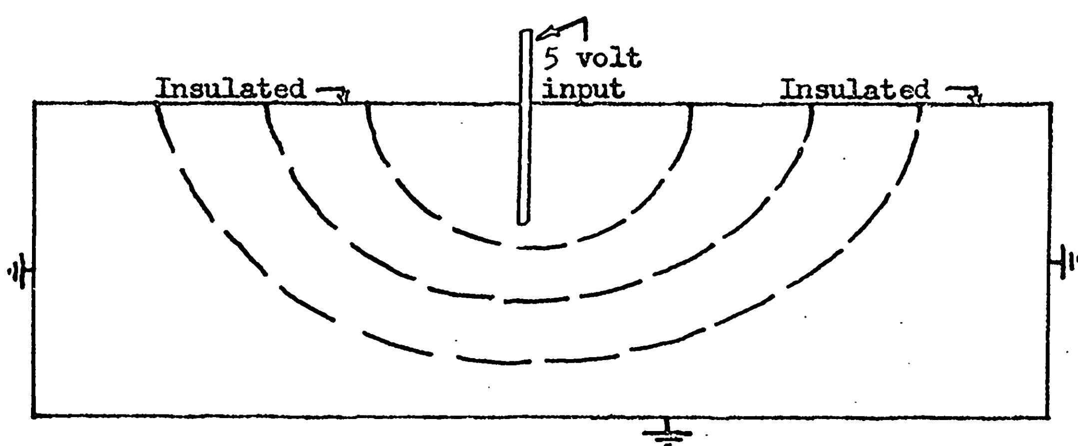


Figure 3

Becker's Model for Flow Contours

The models solved by Becker were mounted on cardboard with rubber cement and the boundaries were painted on with silver conducting paint. Contact was made in a manner similar to that of Peattie. Becker made no allowance for the anistropic properties of the paper, assuming a uniform resistance in all directions. Several types of problems were attempted, starting with a sheet piling in an isotropic soil similar to that worked by Peattie. Becker, however, also solved the flow lines by the conducting sheet by reversing the insulating and conducting surfaces as shown in Figure 3.

The number of flow lines was purposely made equal to the number of equipotential lines, since this now makes the equation for the flow of electricity identical to the equation for the flow of water. The equation for the flow of current through the curvilinear rectangule resulting from making the equipotential and flow lines equal by ohms law must be of the form $q_e = H/R$, where H is the voltage drop and R is the resistance in the direction of the drop. R is equal to rb/a, where r is the resistance of the paper in ohms per square and a and b are as shown in Figure 4. Letting $K_e = 1/r$ and substituting R into the current equation gives $q_e = (a\ K_e\ H)/b$, which is identical in form to the water flow equation $q = (M_f\ K\ H)/M_d$ where Mf/Md is the ratio of the flow paths to the equipotential drops. This ratio is similar to the ratio of a/b and may be found by solving for only one rectangle instead of the whole flow net as shown for the sheet piling problem in Figure 4.

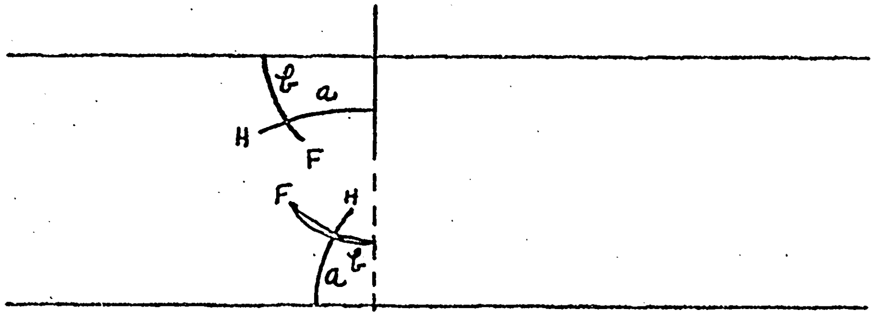


Figure 4

Becker's Curvilinear Rectangle for a/b

Becker has also solved orthotropic soil problems by ruling lines on "Teledeltos" paper with a drawing pen filled with silver conducting paint. By varying the spacing of the lines it was possible to change the resistance of the paper to any ratio. The ratio could not be determined in advance, however, and to get a specific ratio was a trial and error type process. The problem of soils of different permeability was also attempted with a trial and error type process where the soil with the higher permeability was represented by coating the conducting sheet with a dilute solution of silver conducting paint, but it was very difficult to obtain the proper ratio of resistances.

Becker gave no values for the percent of error involved in the various types of solutions he tried; however, he does indicate that the results were in fair agreement with known solutions.

3. SOLUTIONS AND TECHNIQUES INVOLVING EQUATIONS OF THE POISSON TYPE

In Poisson-type equations $\nabla^2 u$ does not equal zero but is equal to some constant. The well-known equation for torsion of a solid prismatic bar is an example of a Poisson-type equation. This equation is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \tag{6}$$

where ϕ is the torsional stress function, G is the modulus of rigidity of the material, and θ is the angle of twist in radians per unit length of the bar. If the distribution of ϕ over the bar cross section is known, then the shear stress is equal to the slope of ϕ at the point interested in and has a direction tangent to the contour of ϕ at the point. This is expressed mathematically as:

$$T = \frac{\partial \phi}{\partial m} \tag{7}$$

where m is the direction normal to the ϕ contours.

Now consider a small elementary area d_y by d_x of a conducting sheet similar to the one shown in Figure 1, only with a distributed current of local density i amperes per unit area fed into the surface (5). Then apply the continuity equation to obtain an equation similar to 1, only equal to $-i\ d_y\ d_x$ instead of zero. After simplifying and applying ohms law, we obtain

$$\frac{P_y}{P_x} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -ip_y \tag{8}$$

where P_x and P_y are again the resistances of the paper in the x and y directions respectively. Again, if $P_x = P_y$, equation 8 reduces to $\nabla^2 V = -i\ P$ and, if $P_x \neq P_y$, we use the same transformation as in Laplace's equation, $X_t = (P_y/P_x)^{1/2}\ x$, to obtain

$$\frac{P_y}{P_x} \frac{\partial^2 \phi}{\partial X_t{}^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \tag{9}$$

The analogy between equations 8 and 9 enables us to use the electrical analogy to solve torsion problems in much the same manner as the familiar soap bubble analogy is used for the same purpose. If the coordinates of the conducting sheet are related linearly to the coordinates of the bar ($x = mX$ and $y = mY$) and $P_y\ i = K\ 2G\theta$, then $V = K\ \phi$ and the shear stress for the bar is given by:

$$T = \frac{m}{K} \frac{\partial V}{\partial N} \tag{10}$$

To solve for the twisting moment again let $V = K\ \phi$ where $K = P_y\ i/2G\theta$ and substitute into the twisting moment equation, $M_t = 2 \iint \phi\ d_y\ d_x$, to give

$$M_t = \frac{4G\theta}{m^2\ iP_y} \iint V\ d_y\ d_x \tag{11}$$

The last part of equation 11 is the volume under the voltage function, so equation 11 may be written as:

$$M_t = \frac{4G\theta}{m^2\ iP_y} \text{(Volume under V function)} \tag{11-A}$$

Warner and Soroka (5) have used the conducting sheet analogy to solve for the stress concentrations of structural angles in torsion, and their technique will be described as it is representative of the technique used on Poisson equations. They used Teledeltos type "L" paper and discovered that, although the resistance of the paper varies with current density, the variation can be held below 0.5% by using a current density of one milliampere/sq. cm., or less. With a current density of 0.8 milliampere/sq. cm., the resistance of the paper was found to be 2350 ohms per square.

The model was mounted on bakelite and, due to symmetry, only half of the angle was used. The free boundaries were established by painting them with silver conducting paint, punching copper conductors into it at 3-inch intervals, and connecting to the negative pole of a 90 volt B battery through a 1000 ohm resistor and a milliammeter. The axis of symmetry was left as an insulated edge. The overall dimensions of the model were 40 inches by 8 inches.

Since it was impractical to apply a distributed current over the entire surface of the sheet, such a current was approximated by feeding an equivalent concentrated current I into a finite 2-inch square grid system. I is determined by multiplying the current density by the area of a square (I = iA) and is fed into the center of gravity of the grid square. At incomplete boundary squares the full concentrated current was fed in and the voltage values reduced by the ratio of the area of the incomplete square to a full square. The feed in contact was made with a one-quarter inch diameter copper disk pressed into the paper with at least 600 psi.

The voltage was measured at the grid intersection points using a dull-pointed brass stylus pressed manually into the paper. Thin copper wires permanently punched into the paper and sealed with a drop of silver paint were also tried with no significant difference in the readings.

The current fed in was adjusted to a high degree of precision by comparison to the output of a standard cell through a 1000 ohm resistor. The voltage measurements were also made to a high degree of precision with a galvanometer.

The experimental results of the ratio of the shear stress at the fillet to the shear stress in the arm were compared to soap bubble solutions and also to a relaxation solution. The conducting sheet solution compared very closely with the relaxation solution and was near the average of the soap bubble solutions which deviated considerably.

A different technique to torsion problems was applied by investigators at the University of California (10). In this work, the torsional stress function was transformed from a Poisson to a Laplace equation by letting $\phi = F - (1/2G\theta)\ (x^2 + y^2)$ to give $\nabla^2 F = 0$, with the condition that $F = (1/2G\theta)\ (x^2 + y^2)$ along the boundary. This is analagous to the conducting sheet with $\nabla^2 V = 0$ and the potential on the boundary of the sheet a function of $(x^2 + y^2)$, namely $F = (1/2\ G\theta)\ (x^2 + y^2)$.

The model studied was a square bar in torsion. The paper used was "Teledeltos" type "H", mounted on pasteboard with rubber cement. The boundary was approximated with electrodes one inch long painted on with silver conducting paint. The applied potentials were in accordance with the boundary function at the center of the electrode. The equipment required for this type of experiment is a power supply for imposing boundary potentials and a measuring circuit. The power supply input was accomplished with several potentiometers in parallel across an a.c. source and the measuring circuit was a potentiometer.

The complete solution required the graphing of the F equipotential lines with $G\theta/2 = 1$ which was assumed in applying the boundary potentials, the transformed equation assumes the form $x^2 + y^2 = F - \phi$. For a constant $F - \phi$, this is the equation of a circle of radius $(F - \phi)^{1/2}$. To evaluate ϕ corresponding to each F, several circles of radius $(F - \phi)^{1/2}$ were plotted over the constant F contours. For the circle $F - \phi = r^2$, and where this intersects F, we find the correct value for ϕ is $F - r^2$ where both F and r^2 are known at the intersection point.

The error in the experimental values compared to the mathematical was found to be 2%. This method has the advantage of not requiring feed in at grid points on the sheet, but it requires elaborate input equipment to adjust the boundary values and it also requires a partial graphical solution.

The only material the author has been able to find on the conducting analogy for the solution of twisting moment on a section in torsion is an unpublished laboratory experiment conducted by Furr (11). In this experiment a rectangular 2-inch by 4-inch bar was represented with a 6-inch by 12-inch model of "Teledeltos" type "H", paper mounted on one-half inch sanded plywood. Brass rods (one-tenth inch diameter) were stapled along the boundaries of the rectangle with contact to the paper made by applying silver conducting paint between the rod and the paper. Rectified 110 A.C. was used to provide D.C. input current which was passed through a 20,000 ohm resistor to give 10 milliamperes of current which was fed into the paper through a 0.561 inch diameter brass disc with a pressure of approximately 4 psi.

The rectangle was laid out with a one-inch grid. The boundary was grounded with current being fed in at the grid intersections with the brass disc while readings of voltage were taken at all grid intersections for each fed-in setting with a blunt brass probe. The equipment used for taking readings was a Weston D.C. milliameter for input current, and an RCA volt ohmyst to measure potential.

The experimental results for shear stress were 11% high and the experimental results for twisting moment was 21% high. This author believes most of this is due to the feed-in probe being placed on the grid intersections instead of at the center of gravity of the grid square, since it is not possible to get the voltage at the grid square with the feed-in probe on it. As this is one of the highest readings, it will have a large effect on the answers.

The author has attempted in the preceding review to give only a brief summary of the theory, technique and application of the conducting sheet analogy. It was undesirable to include many of the details of the solutions, and the reader is referred to the references for further information on a specific type problem. To the best of the author's knowledge, this review has covered all the significant work on the conducting sheet analogy that has been published to date.

III. EXPERIMENTAL PROCEDURE AND PROBLEMS

1. THE PAPER AND MOUNTING

The first phase of the work was to determine a suitable backing and method for mounting the "Teledeltos" paper. This was done by cutting several long narrow strips of "Teledeltos" (23" x 1") and mounting them on various backings with both rubber cement and a good quality glue. Some of the strips were transverse and some longitudinal to the direction of the roll of the paper so that the directional difference in resistances could also be determined.

The three types of backings used were: 1/4" plywood, 1/4" sanded and varnished plywood and 1/4" plexiglass. Contact was made at each end of the paper strips by painting a strip of silver conducting paint (General Cement #21-1) across the ends so that the final length of all strips was 22". Contact was made to the silver paint with a copper wire attached with a drop of solder.

The values of the resistances were found to vary considerably with the type of backing and also with the type of glue. The ratio of transverse to longitudinal resistance, however, was found to be independent of the backing and the glue. The values of this ratio were all about 1.10 with the transverse resistance the higher. The fact that the resistance is dependent on the mounting conditions means that for Poisson type equations the resistance per square must be measured on a square mounted in the same manner and on the same material as the model.

The plywood backing was found to be unsatisfactory since the pickup probe caused damage to the paper when a firm pressure was applied. The plexiglass was found to be very satisfactory in this respect and also gave a very smooth background. The rubber cement seemed to give better mounting results than the glue. The glue mounted strips had many small pockets of air entrapped causing bubbles while the rubber cement mounted strips were smooth and had no bubbles. It was found also that the rubber cement mounted strips were easily removed from the plexiglass after the tests were run, thus enabling the same plexiglass to be used over and over again. The glued strips were much more difficult to remove. In view of these facts it was decided to use plexiglass and rubber cement (Carter #845) for all future experimental work.

2. THE HEAT FLOW PROBLEM

In the second phase of the work a Laplace type problem was used since the input and ground connections are permanent, making it possible to investigate pickup probe values with other circuit conditions relatively constant. It was decided that this was also the best type of problem to check the error between voltmeter and potentiometer readings.

The problem selected was a heat flow problem, similar to one solved by Furr (6). The problem was a semi-infinite strip of width π with the sides held at zero temperature and the end held at a constant temperature K. At this point four 8" x 8 1/2" calibration models were cut. Two were cut with the 8 1/2" side parallel to the edge of the paper and two with the 8 1/2" side perpendicular to the edge of the paper. They were then mounted with rubber cement on plexiglass in such a manner that the resistance ratio of transverse to longitudinal resistance could be measured with the light side and also with the dark side up. The end of the models were painted with silver paint strips 1/4" wide to leave 8" x 8" squares. Copper wires were connected with a drop of solder and the resistances were measured with an ohmmeter (Simpson Model 303) and found to be the same for the light and dark side. The values were 1950 ohms for the transverse resistance and 1780 for the longitudinal resistance to give a ratio of 1.095 or 1.10 to three figure accuracy, which agreed with the ratio taken on the 22" strips. The resistances were also measured with a potentrometer and standard cell with the D.C. input current adjusted to a high degree of precision with the circuit shown in Figure 5. The values of the resistance with this setup were 1975 ohms for the

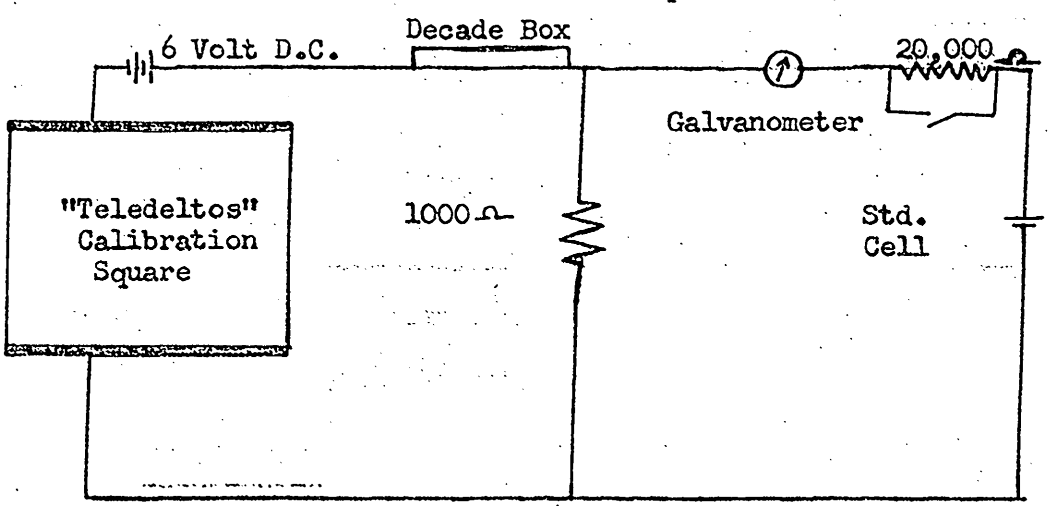


Figure 5

Standard Cell Circuit for Current Input

transverse and 1803 ohms for the longitudinal to give a ratio of 1.095 or 1.10 to three figures.

The dimensions of the heat flow problem were then changed to eliminate the effect of the unequal resistance of the paper. The transformation used was $x_t = x \left(\frac{py}{px}\right)^{1/2}$ and with the y axis coinciding to the transverse direction of the paper, $x_t = 1.05\ x$.

Two heat flow models were set up on the same sheet of plexiglass, one with the dark side of the paper up and the other with the lighter aluminum side up, to determine any differences in the two surfaces of the paper. The models were cut with a 1/4" border which was painted with silver conducting paint. Copper wires were attached with a drop of solder at3 inch intervals along the ground and input boundaries. The resistance of the silver paint boundaries was checked and found to be less than one ohm between the copper wires. Compared to the resistance of the paper this was negligible. A sketch of the model is shown in Appendix B.

The circuit diagrams for the D.C. potentiometer, D.C. voltmeter and A.C. voltmeter circuits are shown in Figures 6, 7 and 8 respectively. Photographs of the instruments and some of the models are shown in Figures 9-12.

The input circuit of Figure 6 consisted of a 6 volt wet cell storage battery, a Simpson Model 373 milliammeter (0-1000 ma), and a Heath Company Model DR-1 decade resistance box (0-100,000 ohms). The measuring circuit was a standard potentiometer setup with a Leeds and Northup Student Potentiometer (0-1.5 volt) and a 6 volt wet cell energizing battery through a DR-1 decade box. The galvanometer and

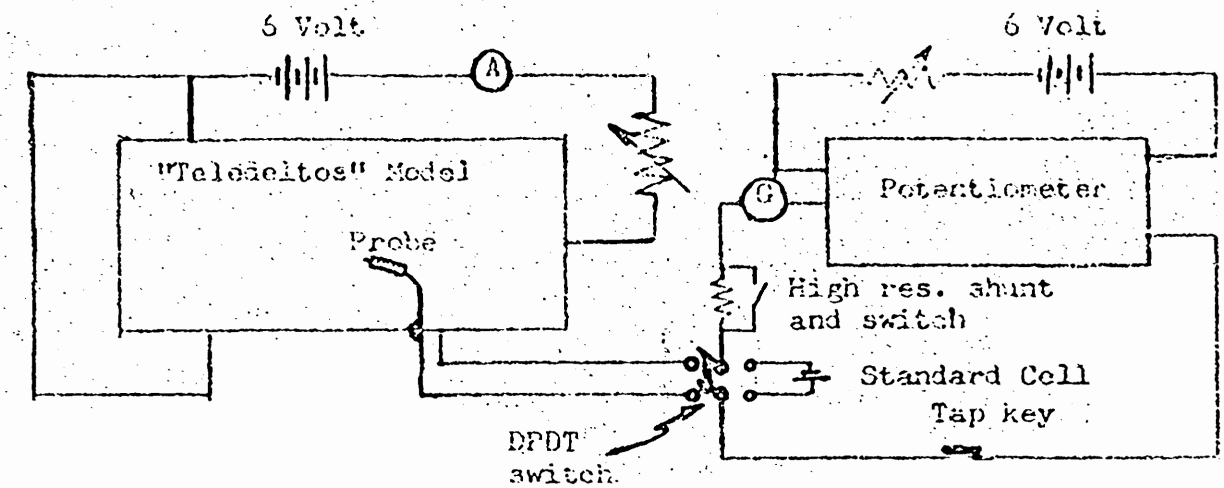


Figure 6

D.C. Potentiometer Circuit

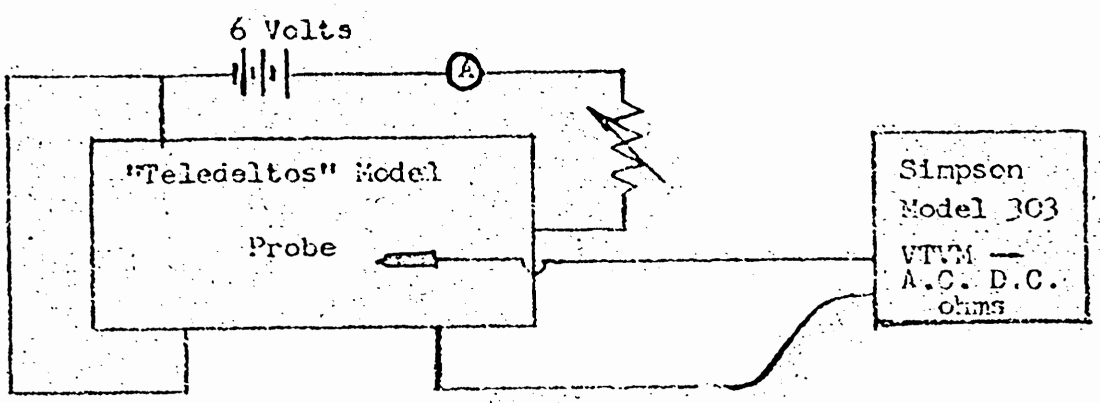


Figure 7

D.C. Voltmeter Circuit

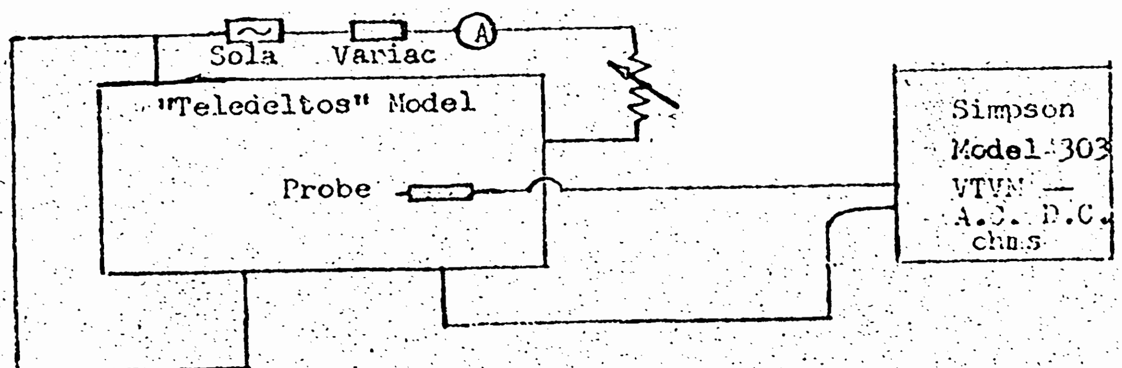


Figure 8

A. C. Voltmeter Circuit

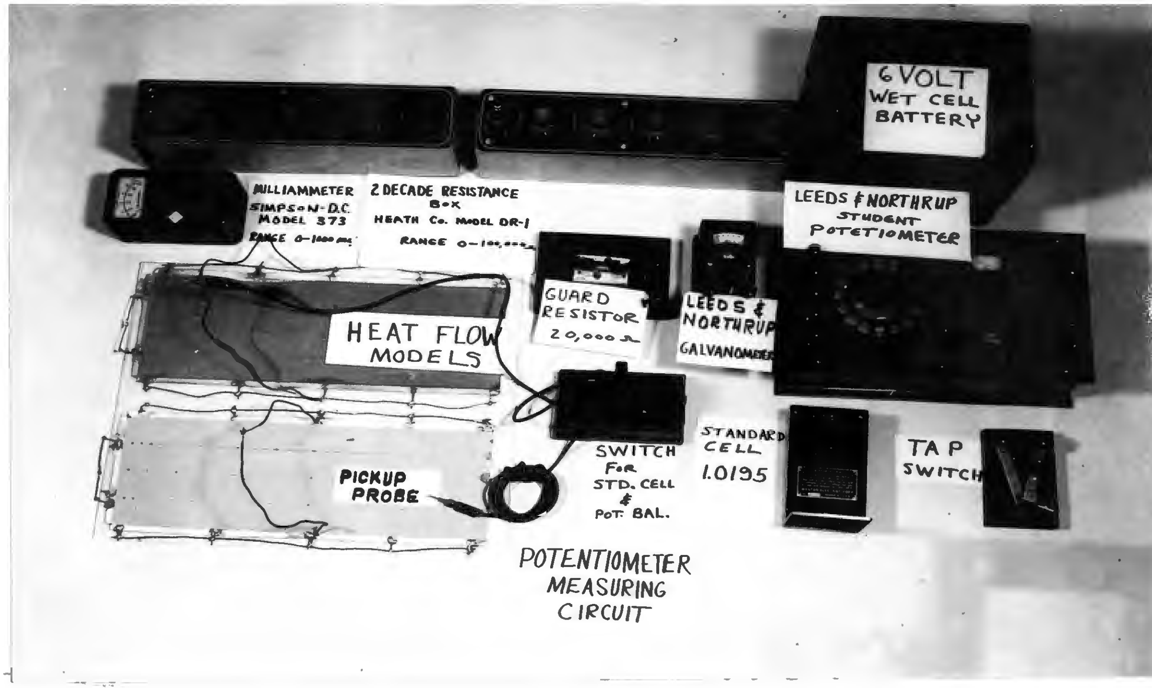


Figure 9

D.C. Potentiometer Measuring Circuit

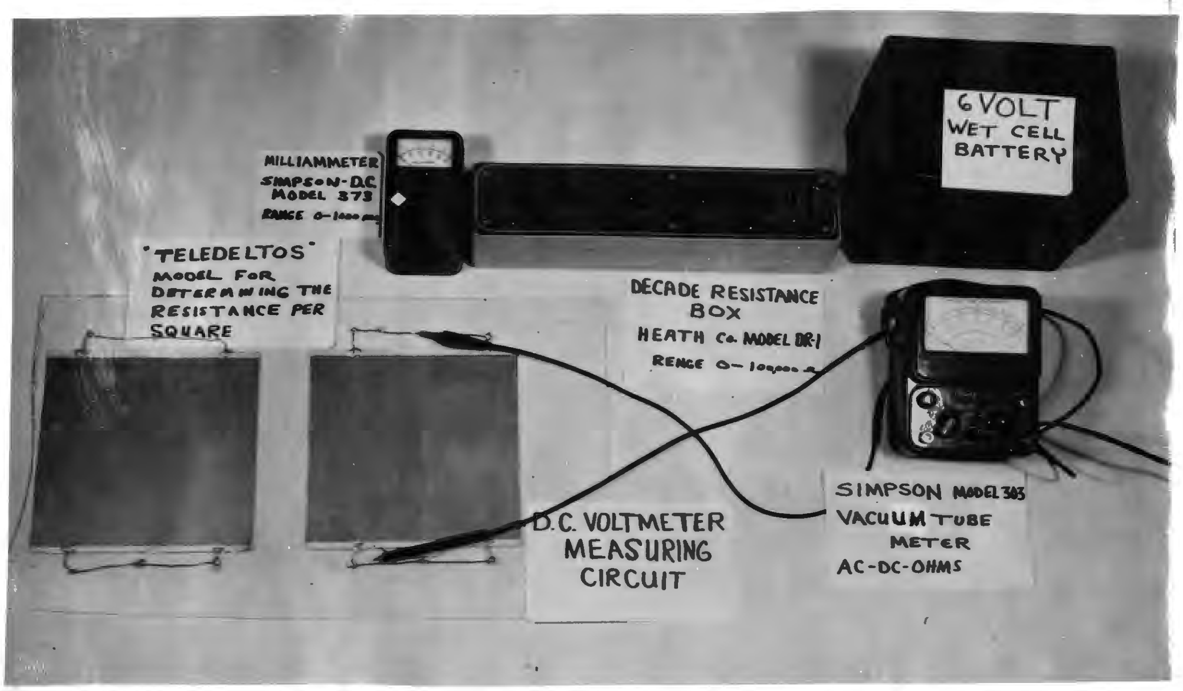


Figure 10

D.C. Voltmeter Measuring Circuit

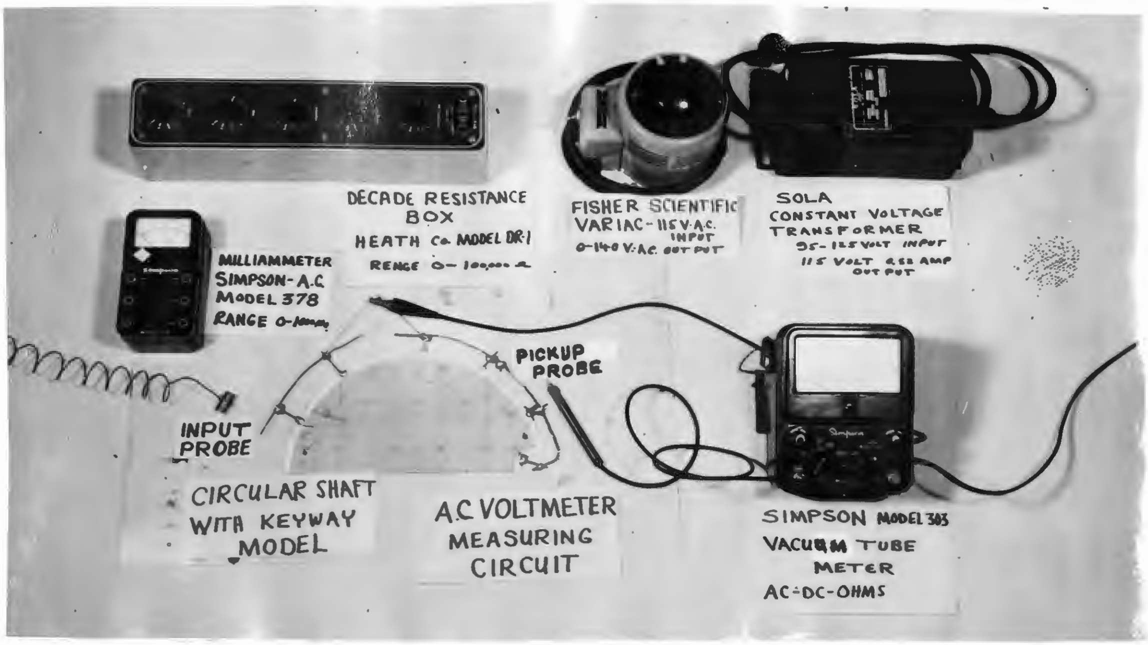


Figure 11

A.C. Voltmeter Measuring Circuit

Figure 12

The Probe Showing the Method of Applying Pressure

1.0195 volt standard cell were also Leeds and Northup. The shunt resistance (20,000 ohms) in parallel with the switch was used to avoid damage to the galvanometer. When balancing the circuit the switch was left open and after the circuit had been roughly balanced the switch was closed to obtain a more sensitive balance. The double pole-double throw switch was very convenient for checking the potentiometer balance and allowed the same galvanometer circuit for potentiometer balancing and measuring the voltage.

The circuit shown in Figure 7 for the D.C. voltmeter measurements had the same input circuit to the model as used with the potentiometer. The measuring circuit consisted merely of a Simpson Model 303 VTVM set to the D.C. voltage scales.

The A.C. circuit shown in Figure 8 has a Sola constant voltage transformer (95-125 V input - 115 V output) from a wall plug as its source. The Fisher Scientific (115 V input, 0-140 V output) variac was used for convenience as a variable resistor and the DR-1 decade box was used for fine adjustment of input voltage. The milliammeter in the line was a Simpson Model 378 (0-1000 ma). The measuring circuit consisted of the Simpson Model 303 VTVM set to the A.C. voltage scales.

The mathematical values of the measured points are given in Appendix A and the experimental values are tabulated in Appendix B. A comparison of the percent of error shows that the D.C. and A.C. voltmeter gave results that compare favorably with those of the potentiometer except for relatively low values of voltage. It was later found that the low voltmeter scales are out of adjustment. If these scales were recalibrated the values in the low range would be much closer to those read with the potentiometer. The voltmeter values were generally within one or two percent of the potentiometer readings. All readings were fairly close to the mathematical values, except toward the end of the model where the values varied from the mathematical solution since the strip was not really semi-infinite as assumed.

There was no significant difference in the voltage values taken on the light and dark models; however, the light side required less pickup probe pressure to make good contact between the base paper and the probe. The values at grid points were found to be consistant, provided the probe was held vertical and a firm manual pressure applied. The probe used was the standard blunt nosed voltmeter probe that came with the Simpson Meter. The voltage at several points was rechecked frequently with the voltage readings remaining the same.

As a result of the heat flow data it was decided to use the light side of the paper for future work and that a manual pickup probe would be entirely satisfactory for taking potential readings. It was also evident that with the voltmeter, better accuracy could be obtained with fairly high voltages. Since a D.C. voltage source above 12 volts was not readily available it was decided to use a 110 volt A.C. source for most of the remaining work. It is generally accepted that better accuracy is attainable with D.C. instruments than with A.C. so it was felt that if the A.C. problems were reasonably accurate, a D.C. setup would give equally as good and probably better results.

3. THE SQUARE BAR IN TORSION PROBLEM

The solution of a square bar in torsion problem was chosen because the exact mathematical solution was available. Also comparison was possible with Warner and Soroka (5) who worked this problem with a precision potentionmeter circuit and obtained results 4.3% below the mathematical answer for the maximum shearing stress.

A. THE MODEL AND PROCEDURE

The model was cut full size to represent a 16" square bar, however, due to symmetry only one quadrant of the bar needed to be represented. With the y axis coinciding with the transverse direction of the paper, the values in the x direction were multiplied by 1.05 to allow for the anistropic resistance of the paper. The 8" by 8.4" section was cut from the paper with an extra 1/4" border on two sides to allow for the silver paint electrodes. The other two sides representing axis of symmetry were left insulated. The grounded outside edges assured a constant value of the function on the outer boundary which is one of the boundary conditions of the torsional ϕ function.

The 1/4" borders were painted leaving an 8" x 8.4" figure and copper wires were soldered at 2 1/2" intervals and connected to one side of the A.C. input circuits. The circuits used were similar to those of Figures 7 and 8, with the exception that the input current was not a parmanent boundary but was fed in with a brass probe. A one inch grid system was laid out over the square with the x dimension actually 1.05" on the model.

The feed in probe was placed at the center of gravity of the grid section, a pressure applied to insure contact with the paper, and the input current was adjusted with the decade box to obtain the proper value. The voltage was then read at all other grid intersection points and the process repeated with the input probe on a new grid section until current had been fed in at all sections. The sum of these voltages at each grid intersection gives the height of the voltage function at that point. To obtain an accurate value of the maximum slope which is known to occur on the outer edge of the axis of symmetry, the voltages were taken in 1/10" increments from the grounded edge along the insulated edge.

B. FEED IN PROBE SIZE AND PRESSURE

Prior to any actual solutions the effect of feed in probe size and pressure was determined. The feed in probe pressure was applied with various size weights on various diameter input probes. It was found that the larger the pressure applied the better the contact was with the paper and the higher the input current for a constant external resistance. To study the effect of this pressure, the external resistance was adjusted to give a constant predetermined value for the input current (10 ma) for different pressures. The voltage was then read and found to be the same at the grid intersections regardless of the input pressure or probe size providing this pressure was large enough to insure uniform contact of the probe to the paper. The minimum pressure seemed to be about 20-30 psi. The test probes used were round brass, with flat smooth ends, varying in diameter from 0.12 to 0.40" and 1/2" long.

Since there was no difference in the voltages obtained it was decided to use the largest probe (0.40" diameter) for future readings, as it was more stable geometrically than the smaller diameter probes. The weight used with this probe was 5 pounds to give a pressure of approximately 40 psi. This combination of probe and pressure was used for all successive readings and problems.

C. THE SIZE OF THE GRID

The size and shape of the grid system are entirely arbitrary and dependent on the individual problem. A square grid system seemed to be the easiest and most convenient to work with and was used. The size of the grids determines the accuracy and speed of solution. A system of very small grid squares will give a more accurate solution than a system of larger ones but it will also require many more readings.

The volume of the function was approximated by taking the average height of the corners of a grid square and multiplying by the area of the base. There is obviously part of the volume of the function left out. The smaller grid involves less error in this respect than a larger one. Figure 13 shows a typical cross-section and the omitted volume for a grid of x by x and for one x/2 by x/2.

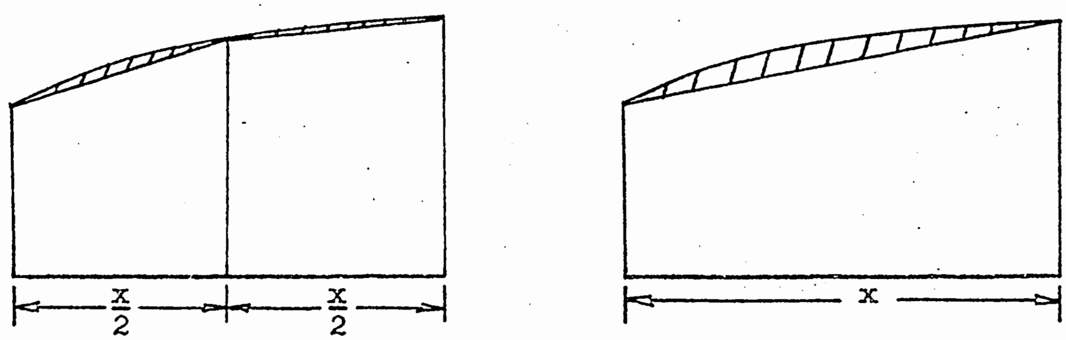


Figure 13

Voltage Function Cross-Sections Showing Neglected Volume

By reducing the input grid from 2" to 1" the error in the solution of the twisting moment for the square bar in torsion was reduced from 11% low to 8% low with the same size pickup grid of 2". The number of voltage readings, however, was increased by a factor of four. If the pickup grid had also been reduced to 1" the number of readings would have increased by a factor of 16 which would have required over 4,000 readings.

The shearing stress was found using a 1" grid input and was in exact agreement with the mathematical solution to three significant figures. It was expected that this would be closer to the true solution than the twisting moment since there is no volume error in determining the slope and the graphing of the three key points allows a very close approximation of the true slope. The graphs for the slope determination are in Appendix B.

No particular size grid system can be used for all problems. The size to be used for a particular problem must be chosen by the individual investigator depending on the accuracy desired and the time available for solution.

D. SUMMARY OF SOLUTIONS

The first solution attempted was with a D.C. circuit for the maximum shearing stress. It was immediately evident that the pickup voltages were very small and difficult to read but the solution was completed. The resulting shearing stress was 39% high which was not surprising in view of the extremely low voltages involved. This was the last D.C. solution attempted in the absence of a high voltage D.C. power supply.

The same problem was then solved using an A.C. circuit. The input current was 10 ma per sq. in. with a one inch grid. The points were summed up and the graph plotted to find the maximum slope of the voltage function. The resistance of the paper was then measured with the same milliammeter and voltmeter that had been used to measure the grid voltages and was found to be 1950 ohms. The resulting maximum shearing stress was in agreement with the mathematical solution, showing no error to these significant figures.

It will be noted that at this point in the work there were marked changes in the weather. The weather had been wet with high humidity and then turned hot with low humidity. The weather continued to fluctuate for several days. This fluctuation is considered important since the measured resistance of the paper varied considerably in the next tests and from day to day.

Part of the change in measured resistance is due to the fact that the voltmeter scales were not consistant with each other, that is the 0-1.2 volt scale read a full 1.2 volts and when the dial was used to switch to the 0-12 volt scale, the same potential on this scale would read only 0.95 volts. This scale difference did not affect the shear stress readings since all readings were read on the same scale, but in the twisting moment problem both scales were used.

The next problem was to solve the twisting moment equation. The feed in grid used was 2" and the pick up grid was also 2" requiring 256 readings for the solution. The input current was 20 ma or a current distribution of 5 ma per sq. in. The resistance of the paper was measured with the 0-12 volt scale on the voltmeter by varying the input currents and using Ohms law to calculate the measured resistance. The resistance was checked at several points over the full range of the scale dial and the values were all within 5 to 10 ohms, indicating that any error in the voltmeter or ammeter was the same over the full range of the dial. The average value of the measured resistance was found to be 1450 ohms. The data and calculations are given in Appendices B and C showing the experimental solution was 11% low.

The problem was reworked with a 1" input grid and a 2" pickup grid requiring 1024 readings. The input current used was 10 ma for a current density of 10 ma per sq. in. The average measured resistance with the 0-12 volt scale was 1450 ohms. The accuracy was increased with only 8% error, again below the mathematical solution.

It seems reasonable in problems involving volume of the voltage function to add a small amount to the experimental solution to compensate for the neglected portion of the function previously discussed and shown in Figure 13. The amount to be added depends on the pickup grid size and for a 2" grid a value of 3% to 5% appears reasonable for this problem. If 5% is added the experimental solutions for torque are only 7% and 3% low for the 2" and 1" input grids respectively. It should be noted that these percentages are only an estimate dependent on the grid size and the shape of the function, but a small increase in the experimental values is certainly justified in any case.

The fact that the lower voltmeter readings were known to vary from the 0-12 volt scale, which was used to measure the resistance of the paper, led to an investigation of the possible effect this would have on the answer. It was found that the total height of the voltage function at a given point was not affected much by the low voltmeter readings. The ordinates never had more than 10% of their height measured with the low scale and it was usually closer to 5%. The higher percentages of error were close to the ground where the ordinates are low and have the least effect on the volume. On the higher ordinates near the center the value was only 1%. The scales were about 20% different so the effect on the final ordinates would be about 20% of 5% which is only about 1%. This error was considered as negligible and no correction was applied to the low voltmeter readings.

At this point it was decided to check the accuracy of the voltmeter. A check with a recently calibrated voltmeter in the Physics Department showed that the 0-1.2 volt scale was reading about 35% high over the entire range. The 0-12 volt scale was also high by about 15% as was the 0-60 volt scale. This discovery illustrated a rather startling fact.. Since the voltages are directly proportional and the resistance of the paper is inversely proportional to the answer, the correct solution can be obtained with a poorly calibrated meter if the resistance of the paper and the voltages are both measured with the same instruments on the same scales.

4. THE CIRCULAR SHAFT WITH CIRCULAR KEYWAY IN TORSION PROBLEM

As a final problem to test the accuracy of the conducting sheet analogy using an ammeter for measuring input and a voltmeter for measuring potential it was decided to work a circular shaft with a circular keyway. The stress function was available for this problem and the mathematical solution is in Appendix A for a 10" diameter shaft with a 1" radius keyway.

The conducting model was laid out to full scale with the dimensions perpendicular to one reference diameter multiplied by 1.05 and the transverse direction of the paper parallel to this diameter. A square grid system was used but of course the transformed value of the distances perpendicular to the reference diameter square was 1.05 times the value in the direction parallel to the diameter. A one inch grid system was chosen which fit into the circle very nicely with a minimum of partial boundary squares. Since the shaft had an axis of symmetry only half of the shaft was represented on "Teledeltos". The outer boundaries were left with a 1/4" border which was later painted with silver conducting paint. The axis of symmetry required no border since it is an insulated edge. A 6" calibrating square was also cut from the same section of the "Teledeltos" roll as the model and was mounted with rubber cement on a piece of plexiglass as was the shaft model. Copper wires were soldered at approximately 5" intervals around the border and the circuit was similar to that used for the square bar in torsion.

The main input and measuring grid size was 2". The input probe was placed at the center of gravity of the grid sections. The input current used was 20 ma or a current density of 5 ma per sq. in. On the partial areas near the circular borders the input probe was placed at the center of gravity of the chosen partial sections and the input current was reduced to the proper value to give a 5 ma/sq. in. current density. This was done by estimating the area of the partial boundary squares as triangles or trapezoids and multiplying 5 ma times the area of a chosen section to obtain the input current values.

The area of the section that was used to feed in the current was also used to measure the volume of the function. The grid system and experimental values are given in Appendix B.

The measured resistance of the paper in the transverse direction was found to be 1610 ohms per square. Since the low voltmeter scales were known to be off, as many readings as possible were taken on the 0-12 volt scale and the resistance of the paper was also measured on this scale.

The percent of error in the shearing stress was 2.2% low and the twisting moment was 4.6% low, with no allowance for the neglected volume. The ratio of M_t to $T_{max.}$ was 2.5% low. In view of the equipment used and the size of the grid system the error is surprisingly small.

It was attempted at this point to determine why the resistance of the paper changed such a large amount over the period of the tests.

As pointed out previously, the relative humidity had changed and the paper is known to change resistance with moisture. The grid square used to calibrate the transverse resistance was wiped on the surface with a slightly dampened cloth. The resistance was measured and found to be over 2,200 ohms/sq. The resistance was periodically checked and as time passed and the paper became drier, the resistance dropped until it eventually approached the original value of 1610 ohms/sq.

It is impractical to obtain a graph of humidity vs. resistance; however, the humidity has an extremely large effect on the resistance of the paper compared to the other variables. By using a resistance calibrating square mounted with the model, the resistance of the paper can be taken under the experimental conditions and this resistance should be used in calculations.

IV. CONCLUSIONS

The original problem was to determine if inexpensive and relatively simple equipment would give fairly accurate results with the conducting sheet analogy. The problems worked show that the simple circuits and inexpensive equipment used for this investigation gave results which were as good, and possibly better, than more complex and expensive circuits used by other investigators. The reason the author believes it is possible to obtain better results than with more precise equipment is the fact that the resistance of the paper varies considerably with changes in the humidity. When precise equipment is used, it takes considerably longer to take the readings so that a problem may be started one day and finished on the next or even several days later. With the voltmeter, readings can be taken comparatively fast and the problem can be completed before the humidity changes appreciably. Of course in controlled humidity conditions, the precise equipment would undoubtedly give better results than the simpler circuit and voltmeter.

The fact that good results were obtained with a voltmeter known to be at least 15% in error illustrates that the technique is very important. By the simple process of using the "apparent" measured resistance of the paper the instrument error is canceled out in Poisson type equations. In Laplace equations the instrument error can be eliminated by adjusting the input to ground potential with the same voltmeter used to take readings. The author feels that the average voltmeter in good adjustment should give better results than were obtained in this investigation since the switching of voltmeter scales would present no difficulties as were encountered in this work.

The results of the problems worked show that inexpensive input circuits and voltmeter potential readings with a calibrating square (for Poisson solutions) will give results with less than 10% error. For shear stresses an accuracy of less than 5% seems reasonable and if an adjustment is made for the neglected portions in calculating the volume of the voltage function an error of less than 5% seems reasonable for torque values. Errors in plotting equipotential lines in Laplace equations should also be less than 5%.

With an adjustment for neglected volumes the accuracy for all type solutions should be within 5% which is sufficient for almost all engineering applications.

APPENDIX A

MATHEMATICAL SOLUTIONS

SOLUTION OF HEAT FLOW PROBLEM

The desired solution is for the steady state temperature distribution in a uniform plate extending from $x = 0$ to $x = d$ and semi-infinite in the positive y direction. The temperature at any point is given by u. The boundary conditions are $u = 0$ on the edges $x = 0$ and $x = d$, $u =$ constant (K) on the edge $y = 0$ and as y approaches infinity, u approaches zero.

The two dimensional steady state heat flow equation is a Laplace equation $\nabla^2 u = 0$ (12) and the solution of this problem with the given boundary conditions is a Fourier series (13) of the form

$$u\,(xy) = \sum_{1}^{\infty} \frac{4K}{n\pi}\, e^{-\frac{\pi n y}{d}} \sin\frac{\pi n x}{d}$$

$$n = 1,3,5$$

If $d = \pi$ this equation reduces to

$$u\,(xy) = \sum_{1}^{\infty} \frac{4\,K}{n\pi}\, e^{-ny} \sin nx$$

$$n = 1,3,5$$

The solution for any value of x and y is now obtained to any desired accuracy by taking the required number of terms in the series. A table on the following page gives the solution for the points used in the experimental work. The values are in terms of K which is a constant temperature applied at the edge $y = 0$.

MATHEMATICAL SOLUTION OF HEAT FLOW PROBLEM FOR u
AT EXPERIMENTAL GRID POINTS

(Answers in items of K allow variation in input)

y	$X = \pi/4 = 3\pi/4$	$X = \pi/2$
.25	.7820 K	.8436 K
.50	.5957 K	.6940 K
.75	.4521 K	.5620 K
1.00	.3448 K	.4488 K
1.50	.2041 K	.2795 K
2.00	.1224 K	.1713 K
3.00	.04484 K	.06341 K
4.00	.01648 K	.02330 K
5.00	.00603 K	.0085 K
6.00	.00225 K	.0031 K

SOLUTION OF SQUARE BAR IN TORSION

The maximum shearing stress and the twisting moment in terms of the shearing modulus of elasticity, G, in psi and the angle of twist, θ, in radians per unit length were taken from tables (14) and are:

$$T_{max.} = 10.8\ G\theta$$

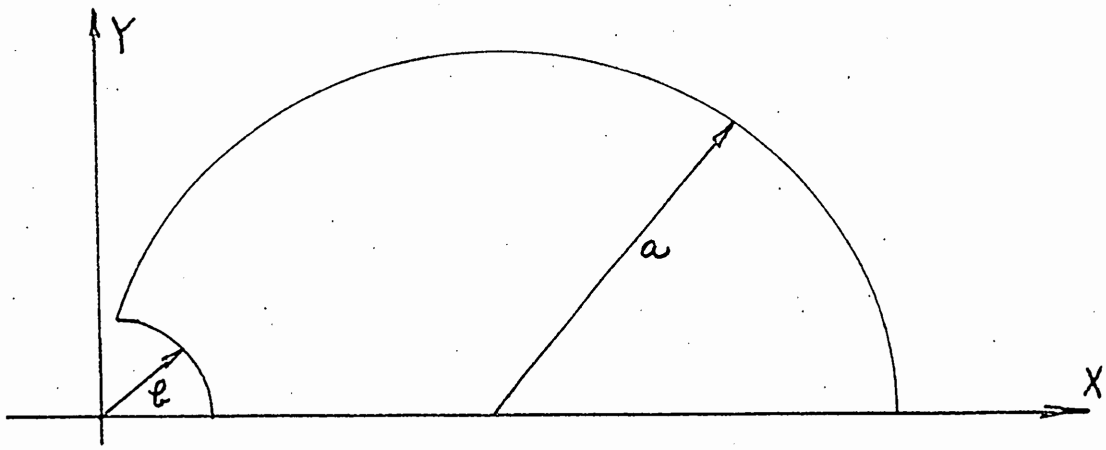
$$M_t = 9{,}240\ G\theta$$

SOLUTION FOR CIRCULAR SHAFT WITH CIRCULAR KEYWAY IN TORSION

The stress function, ϕ, for this problem is given by Den Hartog (15) and is

$$\phi = \frac{-G\theta}{2} \left(x^2 + y^2 - 2\,ax + \frac{2b^2ax}{x^2 + y^2} - b^2\right)$$

where a is the radius of the shaft and b is the radius of the keyway as shown in the sketch below.



CIRCULAR SHAFT WITH KEYWAY

The value of the maximum shearing stress is given by finding the slope of the ϕ function where the maximum shear stress is known to occur. The slope required is $\frac{\partial \phi}{\partial x}$ evaluated at $x = b$, $y = 0$.

$$\frac{\partial \phi}{\partial x} = \frac{-G\theta}{2}\left[2x - 2a + \frac{4ab^2x^2}{(x^2 + y^2)^2} + \frac{2ab^2}{x^2 + y^2}\right]$$

at $x = b$, $y = o$

$$\frac{\partial \phi}{\partial x} = -G\theta\,(b - 2ab^2)$$

for the experimental problem $a = 5$, $b = 1$ and $T_{max.} = -G\theta\,(1-10) =$ $9\,G\theta$.

The value for the twisting moment is given by $M_t = 2\iint \phi$ $d_y\ d_x$. In this problem it is advisable to change to polar coordinates where $x = r \cos \Theta$, $y = r \sin \Theta$. This makes the equation of the shaft $r = 2a \cos \Theta$ and the keyway equation is $r = b$. The upper limit on Θ is the intersection point of the two circles which is $\Theta = 1.4713$ rad. and by symmetry we can make the lower limit zero and multiply by 2. In polar coordinates $d_y\ d_x$ is replaced by $r\ dr\ d\Theta$ and the ϕ function is:

$$\phi = \frac{-G\Theta}{2} \left(r^2 - 2\, a\, r \cos \Theta + \frac{2b^2\, a \cos \Theta}{r} - b^2\right)$$

M_t with $a = 5$ and $b = 1$ is

$$M_t = -2G\Theta \int_0^{1.4713} \int_1^{10 \cos \Theta} (r^3 - 10\, r^2 \cos \Theta + 10 \cos \Theta - 5)\ dr\ d\Theta.$$

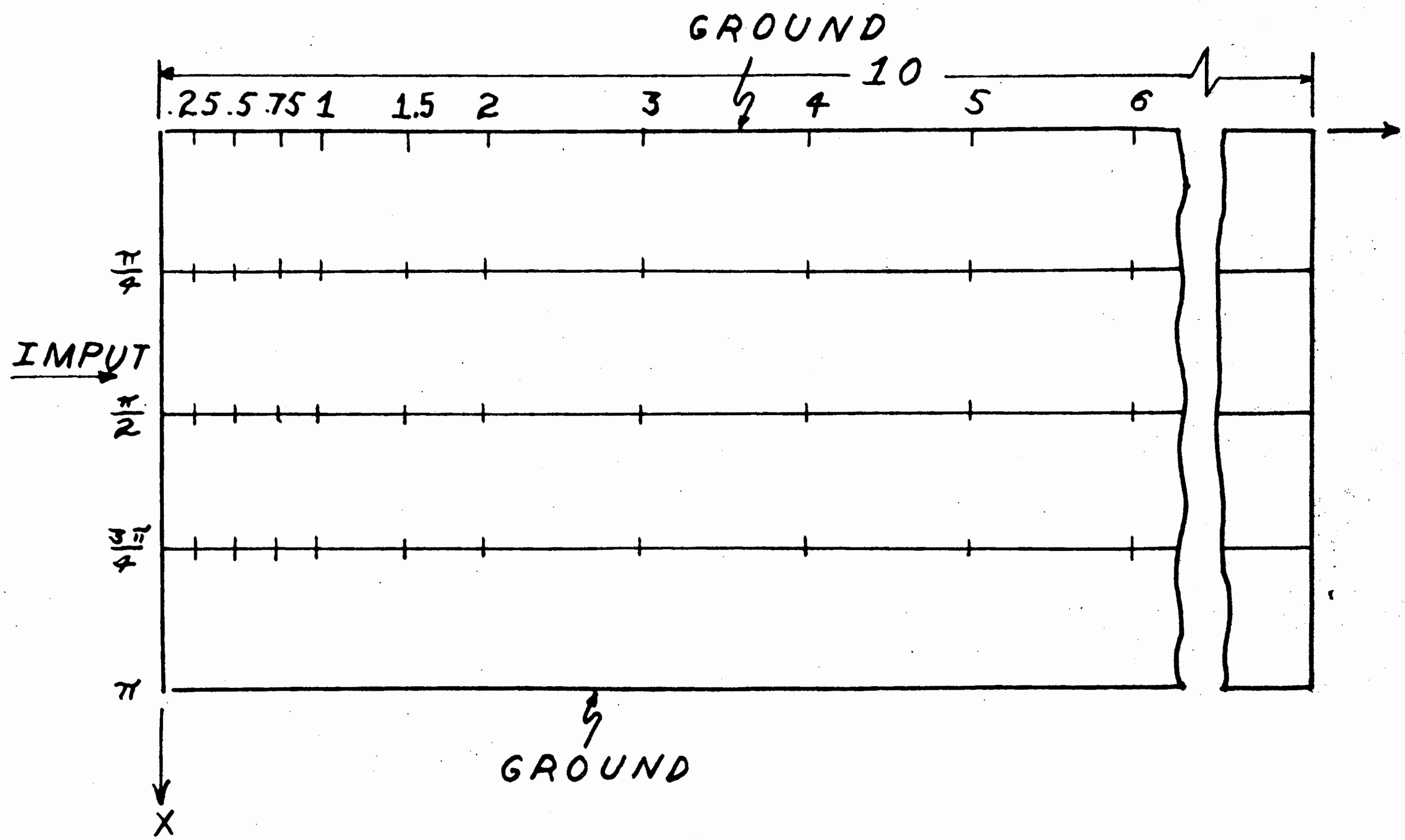
This was evaluated and the result was:

$$M_t = 800\ G\Theta$$

APPENDIX B

EXPERIMENTAL DATA

GRID SYSTEM FOR HEAT FLOW MODEL



METER READINGS AND PERCENT ERROR VALUES FOR HEAT FLOW PROBLEM
DARK SIDE OF PAPER WITH 4 VOLT INPUT

	$X = \pi/4$				$X = \pi/2$				$X = 3\pi/4$			
			Voltmeter				Voltmeter				Voltmeter	
y	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.
.25	3.13	3.13	3.15	3.12	3.17	3.38	3.40	3.38	3.13	3.13	3.13	3.12
.50	2.38	2.39	2.36	2.36	2.78	2.79	2.80	2.78	2.38	2.39	2.40	2.38
.75	1.810	1.84	1.82	1.79	2.25	2.26	2.26	2.25	1.81	1.84	1.82	1.80
1.00	1.380	1.41	1.41	1.39	1.79	1.84	1.85	1.80	1.38	1.41	1.45	1.38
1.50	0.816	0.85	0.850	0.855	1.12	1.15	1.20	1.13	0.816	0.840	0.90	0.85
2.00	0.490	0.510	0.455	0.510	0.685	0.705	0.630	0.685	0.490	0.511	0.45	0.54
3.00	0.180	0.187	0.178	0.200	0.254	0.273	0.240	0.250	0.179	0.190	0.175	0.22
4.00	0.066	0.072	0.061	0.080	0.093	0.104	0.088	0.060	0.066	0.075	0.062	0.080
5.00	0.024	0.026	0.022	0.030	0.034	0.039	0.035	0.015	0.024	0.026	0.023	0.030
6.00	0.010	0.010	0.009		0.013	0.014	0.01	-	0.010	0.010	0.009	-

PERCENT ERROR

y	Pot.	D.C.	A.C.	Pot.	D.C.	A.C.	Pot.	D.C.	A.C.
.25	0	0.6	0.3	0.3	0.9	0.3	0	0	0.3
.50	.4	0.8	0.8	0.4	0.7	0	0.42	0.84	0.0
.75	1.7	0.5	1.1	0.4	0.4	0	0.20	0.60	0.60
1.00	2.1	2.2	0.7	2.9	3.4	0.6	2.11	5.1	0.73
1.50	4.2	3.66	4.2	13.0	7.1	0.9	2.94	11.1	4.9
2.00	4.1	10.8	10	2.9	8.1	2.8	4.1	8.2	10.2
3.00	3.8	7.6	22.2	2.9	4.0	1.6	6.1	7.9	22.9
4.00	9.1	7.6	21.0	11.8	5.4	42	9.4	6.1	21
5.00	8.2	18.3	25.0	14.8	2.9	61	8.3	4.2	25
6.00	-	10	-	7.7	23.0	-	0	10	-

LIGHT SIDE OF PAPER WITH 4 VOLT INPUT

	X = $\pi/4$				X = $\pi/2$				X = $3\pi/4$			
			Voltmeter				Voltmeter				Voltmeter	
y	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.
.25	3.13	3.11	3.13	3.08	3.37	3.33	3.38	3.35	3.13	3.11	3.12	3.08
.50	2.38	2.36	2.38	2.30	2.78	2.75	2.80	2.70	2.38	2.38	2.40	2.30
.75	1.81	1.79	1.810	1.78	2.25	2.23	2.25	2.20	1.81	1.79	1.81	1.79
1.00	1.38	1.37	1.400	1.38	1.80	1.80	1.80	1.76	1.38	1.37	1.40	1.36
1.50	0.816	0.823	0.810	0.802	1.12	1.13	1.10	1.12	0.816	0.823	0.815	0.810
2.00	0.490	0.50	0.490	0.52	0.685	0.691	0.685	0.660	0.490	0.501	0.495	0.52
3.00	0.179	0.189	0.176	0.220	0.254	0.269	0.255	0.290	0.179	0.189	0.179	0.22
4.00	0.066	0.071	0.064	0.07	0.093	0.10	0.094	0.050	0.066	0.071	0.065	0.070
5.00	0.024	0.026	0.023	0.03	0.034	0.048	0.035	0.010	0.024	0.026	0.023	0.03
6.00	0.10	0.010	0.009	-	0.013	0.014	0.0012	-	0.010	0.010	0.009	-

PERCENT ERROR

y	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.	Math	Pot.	D.C.	A.C.
.25		0.64	0	1.61		1.18	0.30	0.60		0.64	0.32	1.16
.50		0.84	0	3.36		1.08	0.73	2.87		0	0.84	3.36
.75		1.15	0	1.66		0.89	0	1.37		1.15	0	1.15
1.00		0.73	1.45	0		0	0	2.22		0.73	1.45	1.46
1.50		0.76	0.74	1.72		0.84	1.79	0		0.76	.10	0.74
2.00		2.25	0	6.10		0.88	0	3.65		2.25	1.13	6.10
3.00		5.57	1.67	22.81		1.69	0.38	13.65		5.57	0	22.81
4.00		7.58	3.03	6.06		7.31	0.86	40.40		7.58	1.52	6.06
5.00		8.34	4.18	25.0		11.75	2.94	73.75		8.34	4.18	25.0
6.00		0	10	-		7.68	7.68	-		0	10	-

DARK SIDE OF PAPER

	Input Voltage (K) = 8 Volts				Input Voltage (K) = 12 Volts			
	$X = \pi/4$ & $3\pi/4$		$X = \pi/2$		$X = \pi/4$ & $3\pi/4$		$X = \pi/2$	
y	Math	Voltmeter A.C.	Math	Voltmeter A.C.	Math	Voltmeter A.C.	Math	Voltmeter A.C.
.25	6.26	6.24	6.75	6.75	9.38	9.35	10.12	10.0
.50	4.77	4.70	5.55	5.55	7.15	7.03	8.33	8.31
.75	3.62	3.60	4.50	4.50	5.42	5.50	6.74	6.65
1.00	2.76	2.75	3.59	3.55	4.14	4.05	5.39	5.38
1.50	1.63	1.60	2.24	2.22	2.45	2.40	3.54	3.41
2.00	0.980	0.990	1.37	1.40	1.47	1.48	2.06	2.05
3.00	0.358	0.355	0.507	0.510	0.537	0.544	0.762	0.780
4.00	0.132	0.105	0.186	0.170	0.198	0.180	0.280	0.270
5.00	0.048	0.030	0.068	0.045	0.072	0.050	0.102	0.080
6.00	0.020	0.010	0.025	0.010	0.030	0.015	0.038	0.018

PERCENT ERROR

y	8 V, $X = \pi/4$ & $3\pi/4$	8 V, $X = \pi/2$	12 V, $X = \pi/4$ & $3\pi/4$	12 V, $X = \pi/2$
.25	0.32	0	0.32	1.01
.50	0.15	0	1.68	0.24
.75	1.12	0	1.48	1.33
1.00	0.36	1.12	2.28	0.19
1.50	1.88	0.89	2.04	3.67
2.00	0.12	2.19	0.68	0.50
3.00	0.84	0.59	1.30	2.36
4.00	20.4	8.60	9.10	3.57
5.00	36.9	33.9	30.6	21.8
6.00	50.0	60.0	50.0	52.7

LIGHT SIDE OF PAPER

y	Input Voltage (K) = 8 Volts: X = π/4 & 3 π/4, Math	Input Voltage (K) = 8 Volts: X = π/4 & 3 π/4, Voltmeter A.C.	Input Voltage (K) = 8 Volts: X = π/2, Math	Input Voltage (K) = 8 Volts: X = π/2, Voltmeter A.C.	Input Voltage (K) = 12 Volts: X = π/4 & 3 π/4, Math	Input Voltage (K) = 12 Volts: X = π/4 & 3 π/4, Voltmeter A.C.	Input Voltage (K) = 12 Volts: X = π/2, Math	Input Voltage (K) = 12 Volts: X = π/2, Voltmeter A.C.
.25	6.26	6.10	6.75	6.60	9.38	9.70	10.2	10.0
.50	4.77	4.65	5.55	5.45	7.15	7.00	8.33	8.20
.75	3.62	3.50	4.50	4.40	5.42	5.30	6.74	6.60
1.00	2.76	2.70	3.59	3.50	4.14	3.95	5.39	5.30
1.50	1.63	1.55	2.24	2.15	2.45	2.38	3.54	3.29
2.00	0.980	0.980	1.37	1.26	1.47	1.40	2.06	1.98
3.00	0.358	0.33	0.507	0.50	0.537	0.520	0.762	0.760
4.00	0.132	0.095	0.186	0.145	0.198	0.160	0.280	0.250
5.00	0.048	0.02	0.068	0.04	0.072	0.040	0.102	0.060
6.00	0.020	-	0.025	-	0.030	-	0.030	0.010

PERCENT ERROR

y	8 Volts, X = π/4 & 3 π/4	8 Volts, X = π/2	12 Volts, X = π/4 & 3 π/4	12 Volts, X = π/2
.25	2.56	2.22	3.41	1.01
.50	2.52	1.82	2.1	2.08
.75	3.31	2.22	2.26	1.67
1.00	2.17	2.50	4.58	7.06
1.50	4.93	3.69	2.86	3.88
2.00	0	8.03	4.85	0.26
3.00	7.87	1.37	19.2	10.72
4.00	28.1	22.0	44.5	41.8
5.00	58.4	41.2	-	-
6.00	-	-	-	-

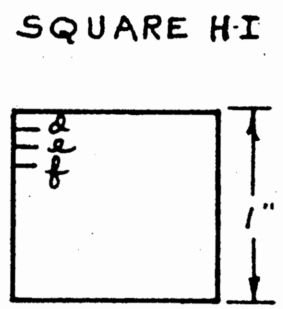
LIGHT SIDE OF PAPER

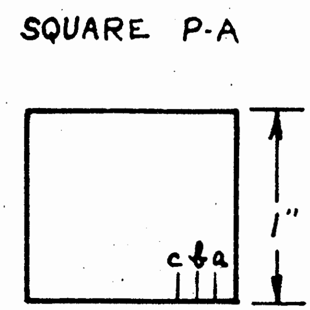
	Input Voltage (K) = 16 Volts				Input Voltage (K) = 20 Volts			
	X = $\pi/4$ & $3\pi/4$		X = $\pi/2$		X = $\pi/4$ & $3\pi/4$		X = $\pi/2$	
y	Math	Voltmeter A.C.	Math	Voltmeter A.C.	Math	Voltmeter A.C.	Math	Voltmeter A.C.
.25	12.51	12.2	13.49	13.30	15.65	15.2	16.85	16.50
.50	9.53	9.65	11.10	11.30	11.90	11.9	13.90	13.80
.75	7.24	7.35	8.99	9.20	9.05	9.30	11.25	11.62
1.00	5.52	5.60	7.18	7.35	6.90	7.10	8.95	9.35
1.50	3.26	3.22	4.47	4.40	4.10	4.18	5.95	5.80
2.00	1.96	1.98	2.74	2.72	2.45	2.50	1.43	3.48
3.00	0.716	0.765	1.01	1.09	0.900	0.980	1.27	1.22
4.00	0.264	0.750	0.372	0.360	0.390	0.330	0.465	0.490
5.00	0.096	0.065	0.136	0.110	0.100	0.090	0.170	0.150
6.00	0.040	0.015	0.051	0.020	0.050	0.020	0.065	0.030

PERCENT ERROR

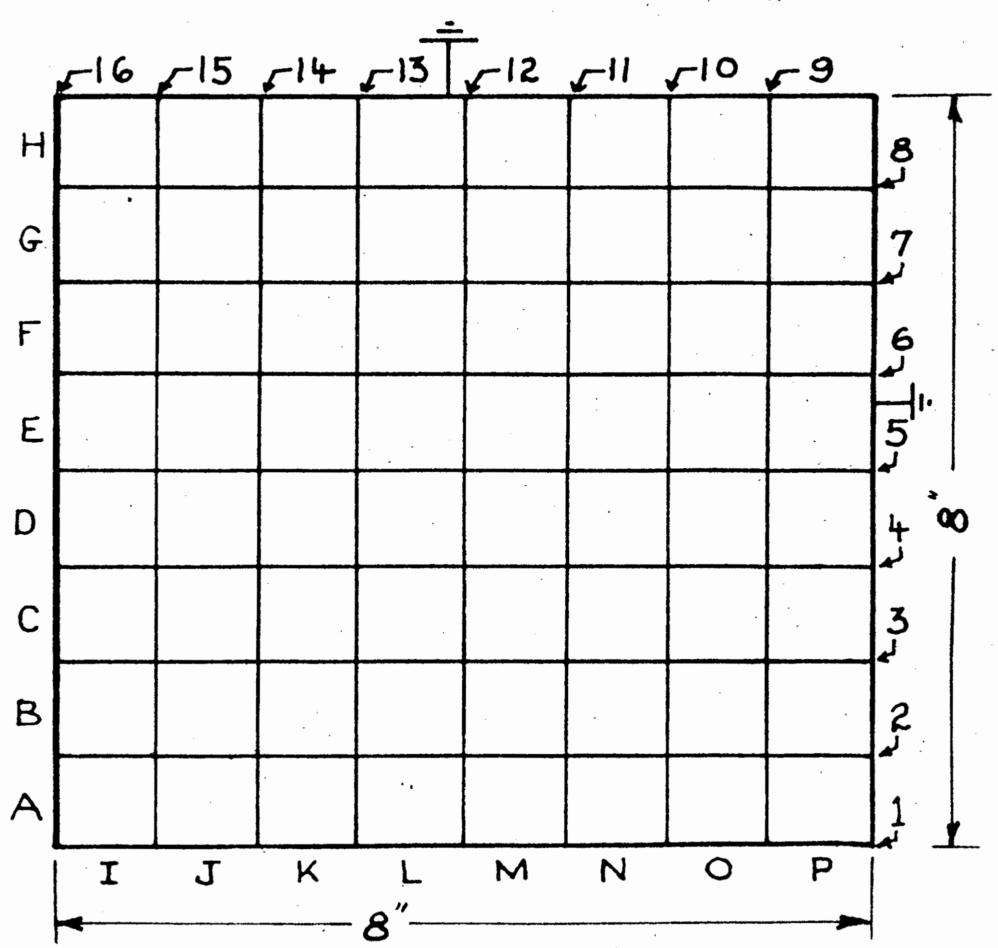
y								
.25		2.47		1.42		2.87		2.08
.50		1.26		1.80		0		0.72
.75		3.80		3.45		2.76		3.38
1.00		1.45		2.37		2.92		4.47
1.50		1.11		1.57		1.95		2.52
2.00		1.03		0.74		2.04		3.50
3.00		6.78		7.92		8.90		3.94
4.00		5.3		3.23		15.40		7.53
5.00		32.3		19.10		10.0		18.0
6.00		62.5		60.8		60		54

GRID SYSTEM FOR SQUARE IN TORSION





NOTE: POINTS a, b, c, AND d, e, f MEASURED AT $\frac{1}{10}$" INCREMENTS FROM GROUNDED EDGE.

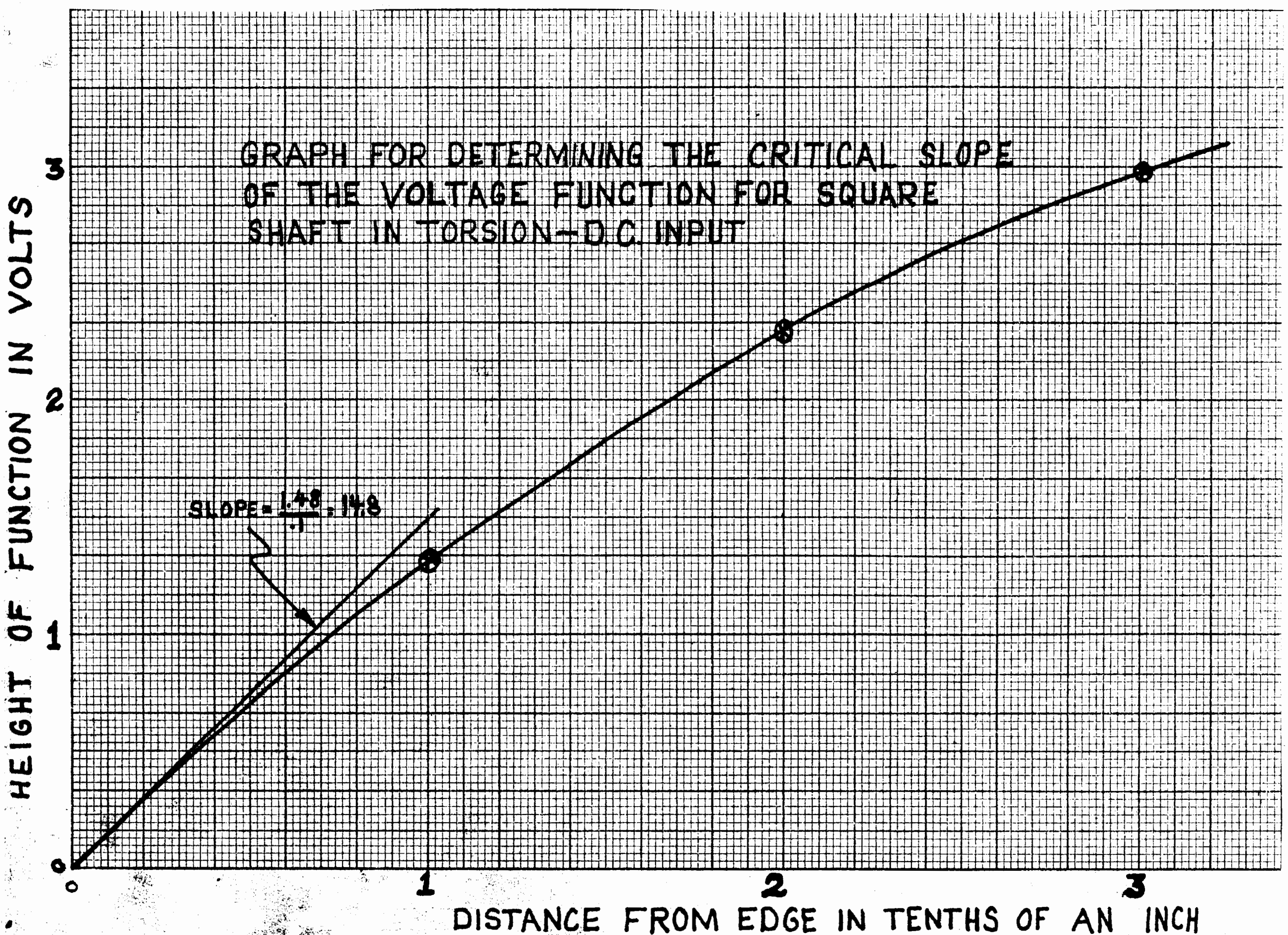


D.C. VOLTMETER READING VALUES FOR MAXIMUM SLOPE FOR SQUARE SHAFT IN TORSION

Feed in Square	Point Numbers a	b	c	Feed in Square	Point Numbers a	b	c	Feed in Square	Point Numbers a	b	c
I-H	0.001	0.001	0.002	I-E	0.018	0.023	0.030	I-B	0.020	0.040	0.057
J-H	0.002	0.004	0.006	J-E	0.018	0.023	0.032	J-B	0.020	0.040	0.060
K-H	0.002	0.004	0.006	K-E	0.010	0.022	0.031	K-B	0.025	0.045	0.063
L-H	0.002	0.004	0.006	L-E	0.010	0.022	0.030	L-B	0.030	0.050	0.070
M-H	0.001	0.001	0.002	M-E	0.010	0.021	0.030	M-B	0.030	0.060	0.080
N-H	0.001	0.002	0.003	N-E	0.010	0.020	0.030	N-B	0.050	0.080	0.100
O-H	0	0	0	O-E	0.012	0.020	0.025	O-B	0.055	0.100	0.125
P-H	0	0	0	P-E	0.005	0.008	0.010	P-B	0.035	0.065	0.090
I-G	0.005	0.008	0.009	I-D	0.018	0.030	0.040	I-A	0.025	0.040	0.055
J-G	0.006	0.010	0.015	J-D	0.019	0.030	0.041	J-A	0.030	0.045	0.060
K-G	0.010	0.014	0.018	K-D	0.015	0.031	0.040	K-A	0.035	0.052	0.065
L-G	0.010	0.014	0.018	L-D	0.018	0.032	0.045	L-A	0.035	0.060	0.080
M-G	0.008	0.012	0.017	M-D	0.020	0.036	0.045	M-A	0.045	0.080	0.100
N-G	0.008	0.014	0.016	N-D	0.018	0.030	0.042	N-A	0.060	0.110	0.140
O-G	0.005	0.008	0.010	O-D	0.015	0.023	0.032	O-A	0.100	0.170	0.225
P-G	0.001	0.001	0.001	P-D	0.010	0.015	0.018	P-A	0.180	0.300	0.380
I-F	0.015	0.020	0.028	I-C	0.018	0.038	0.050				
J-F	0.012	0.019	0.023	J-C	0.020	0.038	0.050				
K-F	0.012	0.019	0.022	K-C	0.020	0.040	0.050				
L-F	0.011	0.019	0.022	L-C	0.025	0.045	0.060				
M-F	0.010	0.010	0.020	M-C	0.025	0.045	0.060				
N-F	0.010	0.016	0.020	N-C	0.030	0.050	0.062				
O-F	0.013	0.018	0.020	O-C	0.030	0.045	0.060				
P-F	0.001	0.002	0.003	P-C	0.016	0.023	0.031				

Summation of Pick Up Points

a	b	c
1.1315	2.289	2.981

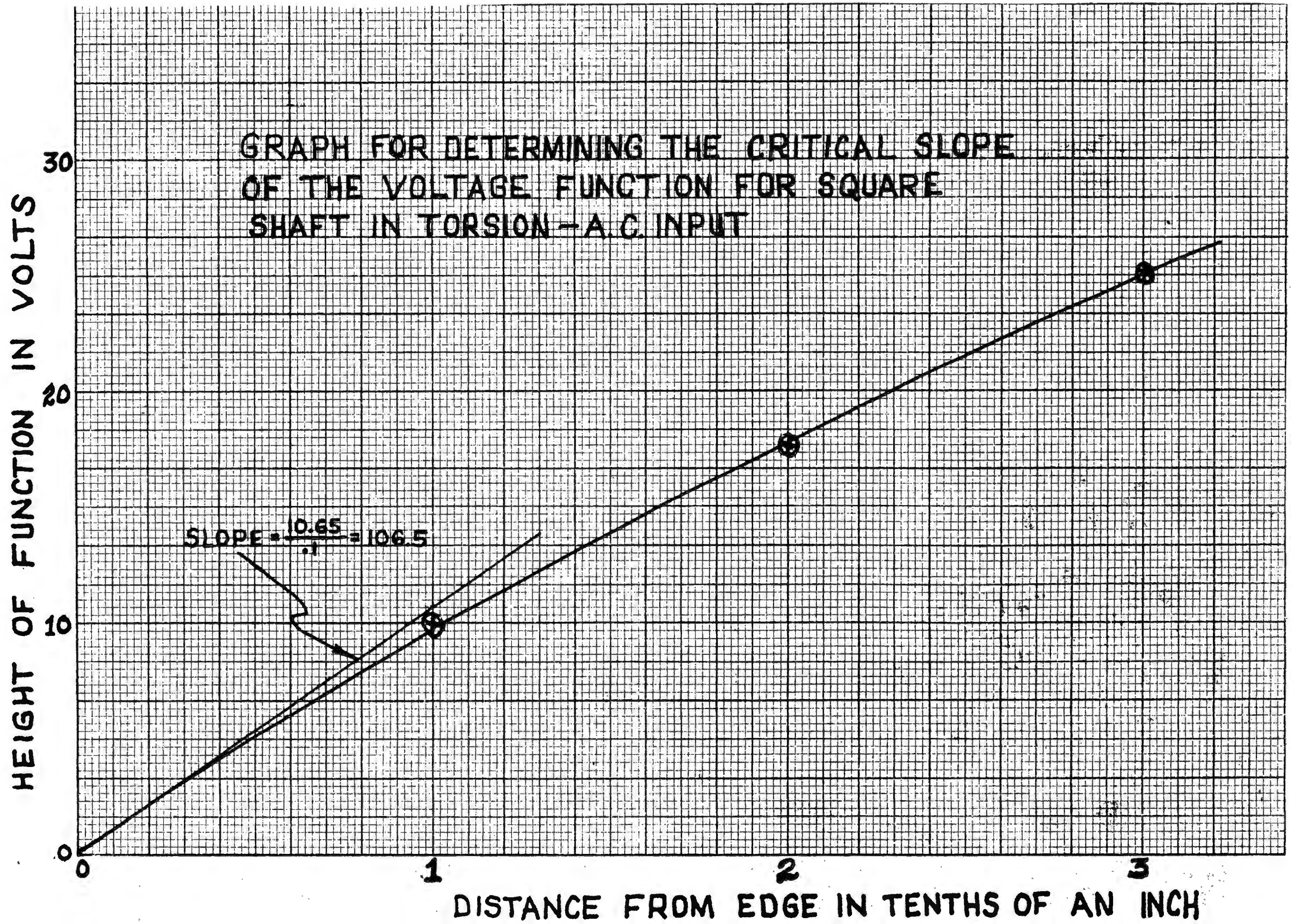

GRAPH FOR DETERMINING THE CRITICAL SLOPE
OF THE VOLTAGE FUNCTION FOR SQUARE
SHAFT IN TORSION—D.C. INPUT
SLOPE = 1.48/.1 = 14.8
HEIGHT OF FUNCTION IN VOLTS
DISTANCE FROM EDGE IN TENTHS OF AN INCH
0
1
2
3
0
1
2
3

A.C. VOLTMETER READING VALUES FOR MAXIMUM SLOPE FOR A SQUARE SHAFT IN TORSION

Feed in Square	Point Numbers d	e	f	Feed in Square	Point Numbers d	e	f	Feed in Square	Point Numbers d	e	f
H-P	0	0	0	H-M	0.011	0.022	0.035	H-J	0.23	0.45	0.63
G-P	0	0	0	G-M	0.060	0.100	0.170	G-J	0.50	0.85	1.16
F-P	0.002	0.004	0.007	F-M	0.090	0.170	0.260	F-J	0.38	0.74	1.01
E-P	0.002	0.005	0.008	E-M	0.110	0.190	0.300	E-J	0.28	0.56	0.80
D-P	0.002	0.005	0.010	D-M	0.100	0.180	0.200	D-J	0.26	0.44	0.63
C-P	0.002	0.005	0.010	C-M	0.100	0.200	0.270	C-J	0.20	0.37	0.57
B-P	0.005	0.010	0.018	B-M	0.100	0.170	0.260	B-J	0.19	0.33	0.48
A-P	0.008	0.015	0.020	A-M	0.080	0.160	0.240	A-J	0.18	0.32	0.47
H-O	0.001	0.001	0.002	H-L	0.040	0.065	0.100	H-I	1.40	2.20	3.00
G-O	0.010	0.018	0.030	G-L	0.130	0.220	0.350	G-I	0.78	1.30	1.75
F-O	0.020	0.040	0.060	F-L	0.160	0.280	0.400	F-I	0.47	0.82	1.19
E-O	0.020	0.040	0.070	E-L	0.180	0.280	0.400	E-I	0.36	0.65	0.90
D-O	0.020	0.045	0.080	D-L	0.170	0.290	0.420	D-I	0.27	0.48	0.70
C-O	0.020	0.045	0.080	C-L	0.160	0.280	0.400	C-I	0.215	0.38	0.58
B-O	0.030	0.050	0.085	B-L	0.150	0.240	0.370	B-I	0.195	0.345	0.50
A-O	0.020	0.060	0.090	A-L	0.120	0.220	0.340	A-I	0.19	0.33	0.48
H-N	0.005	0.010	0.015	H-K	0.080	0.170	0.260				
G-N	0.025	0.040	0.080	G-K	0.260	0.420	0.620				
F-N	0.050	0.050	0.120	F-K	0.300	0.500	0.720				
E-N	0.050	0.100	0.170	E-K	0.230	0.440	0.660				
D-N	0.060	0.110	0.180	D-K	0.220	0.380	0.580				
C-N	0.060	0.110	0.190	C-K	0.200	0.310	0.500				
B-N	0.060	0.100	0.170	B-K	0.170	0.310	0.440				
A-N	0.060	0.100	0.160	A-K	0.150	0.280	0.400				

Summation of Pick Up Points

d	e	f
100.23	174.40	253.00


GRAPH FOR DETERMINING THE CRITICAL SLOPE
OF THE VOLTAGE FUNCTION FOR SQUARE
SHAFT IN TORSION – A.C. INPUT
SLOPE = 10.65 / .1 = 106.5
HEIGHT OF FUNCTION IN VOLTS
DISTANCE FROM EDGE IN TENTHS OF AN INCH
0
10
20
30
1
2
3

VOLTMETER READINGS FOR DETERMINING THE VOLUME OF THE VOLTAGE FUNCTION
FOR SQUARE SHAFT USING A 2 INCH INPUT GRID AND A 2 INCH PICK UP POINT GRID

Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.
9-2	10-1	6.6	9-4	12-1	1.0	9-8	14-5	0.53
"	12-1	3.9	"	14-1	0.8	"	16-5	0.46
"	14-1	2.9	"	16-7	0.8	"	10-7	2.2
"	16-1	2.6	9-6	10-1	1.0	"	12-7	0.80
"	10-3	4.3	"	12-1	1.45	"	14-7	0.4
"	12-3	3.3	"	14-1	1.45	"	16-7	0.31
"	14-3	2.6	"	16-1	1.4	11-2	10-1	11.4
"	16-3	2.3	"	10-3	1.4	"	12-1	13.1
"	10-5	1.7	"	12-3	1.65	"	14-1	09.9
"	12-5	1.9	"	14-3	1.50	"	16-1	08.7
"	14-5	1.8	"	16-3	1.35	"	10-3	08.7
"	16-5	1.7	"	10-5	3.6	"	12-3	11.0
"	10-7	0.68	"	12-5	2.0	"	14-3	08.6
"	12-7	0.95	"	14-5	1.35	"	16-3	07.8
"	14-7	0.95	"	10-5	1.2	"	10-5	03.7
"	16-7	0.88	"	10-7	3.2	"	12-5	05.8
9-4	10-1	2.2	"	12-7	1.5	"	14-5	05.8
"	12-1	2.4	"	14-7	0.9	"	16-5	05.4
"	14-1	2.1	"	16-7	0.84	"	10-7	1.6
"	16-2	2.0	9-8	10-1	0.28	"	12-7	2.5
"	10-3	3.5	"	12-1	0.45	"	14-7	2.8
"	12-3	2.3	"	14-1	0.51	"	16-7	2.6
"	14-3	2.0	"	16-1	0.50	11-4	10-1	5.0
"	16-3	1.8	"	10-3	0.38	"	12-1	6.4
"	10-5	3.4	"	12-3	0.52	"	14-1	7.2
"	12-5	2.0	"	14-3	0.52	"	16-1	6.8
"	14-5	1.6	"	16-3	0.50	"	10-3	6.5
"	16-5	1.4	"	10-5	0.78	"	12-3	8.8
"	10-7	1.0	"	12-5	0.70	"	14-3	7.1

Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.
11-4	16-3	6.3	11-8	10-5	1.15	13-4	12-5	9.4
"	10-5	6.3	"	12-5	1.75	"	14-5	11.8
"	12-5	7.8	"	14-5	1.14	"	16-5	10.1
"	14-5	5.8	"	16-5	1.30	"	10-7	2.1
"	16-5	5.1	"	10-7	2.5	"	12-7	4.2
"	10-7	2.3	"	12-7	3.2	"	14-7	5.2
"	12-7	3.2	"	14-7	1.3	"	16-7	5.0
"	14-7	3.0	"	16-7	0.9	13-6	10-1	2.7
"	16-7	2.6	13-2	10-1	7.4	"	12-1	5.0
11-6	10-1	2.4	"	12-1	16.9	"	14-1	6.4
"	12-1	3.25	"	14-1	20.8	"	16-1	6.7
"	14-1	4.15	"	16-1	17.7	"	10-3	2.9
"	16-1	4.2	"	10-3	6.4	"	12-3	5.7
"	10-3	2.8	"	12-3	13.0	"	14-3	7.0
"	12-3	4.4	"	14-3	16.0	"	16-3	7.1
"	14-3	4.35	"	16-3	15.0	"	10-5	3.1
"	16-3	4.15	"	10-5	3.8	"	12-5	7.6
"	10-5	5.1	"	12-5	7.6	"	14-5	8.7
"	12-5	6.4	"	14-5	9.8	"	16-5	7.5
"	14-5	4.4	"	16-5	9.7	"	10-7	2.3
"	16-5	3.7	"	10-7	1.9	"	12-7	6.3
"	10-7	4.3	"	12-7	3.4	"	14.7	7.0
"	12-7	5.5	"	14-7	4.3	"	16-7	5.2
"	14-7	3.0	"	16-7	4.8	13-8	10-1	0.88
"	16-7	2.4	13-4	10-1	5.0	"	12-1	1.3
11-8	10-1	0.71	"	12-1	9.8	"	14-1	1.8
"	12-1	1.0	"	14-1	12.0	"	16-1	1.9
"	14-1	1.2	"	16-1	12.1	"	10-3	0.94
"	16-1	1.2	"	10-3	05.0	"	12-3	1.5
"	10-3	0.85	"	12-3	11	"	14-3	1.95
"	12-3	1.1	"	14-3	13.1	"	16-3	2.0
"	14-3	1.25	"	16-3	12.0	"	10-5	1.10
"	16-3	1.25	"	10-5	4.0	"	12-5	2.0

Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.
13-8	14-5	2.50	15-4	14-5	13.6	15-8	14-5	3.6
"	16-5	2.4	"	16-5	16.6	"	16-5	4.6
"	10-7	1.08	"	10-7	1.80	"	10-7	0.74
"	12-7	3.15	"	12-7	4.0	"	12-7	1.8
"	14-7	3.8	"	14-7	6.45	"	14-7	5.1
"	16-7	2.4	"	16-7	7.60	"	16-7	7.8
15-2	10-1	5.80	16-6	10-1	26.5			
"	12-1	12.6	"	12-1	5.5			
"	14-1	23.5	"	14-1	7.9			
"	16-1	32.5	"	16-1	8.8			
"	10-3	5.30	"	10-3	2.60			
"	12-3	11.7	"	12-3	5.7			
"	14-3	19.3	"	14-3	8.7			
"	16-3	25.0	"	16-3	10.0			
"	10-5	3.65	"	10-5	2.35			
"	12-5	7.80	"	12-5	5.60			
"	14-5	11.5	"	14-5	10.9			
"	16-5	13.0	"	16-5	14.5			
"	10-7	1.85	"	10-7	1.50			
"	12-7	3.80	"	12-7	3.75			
"	14-7	5.40	"	14-7	8.5			
"	16-7	5.9	"	16-7	12.0			
15-4	10-1	4.5	15-8	10-1	0.100			
"	12-1	9.6	"	12-1	1.8			
"	14-1	14.5	"	14-1	2.6			
"	16-1	16.9	"	16-1	2.9			
"	10-3	4.2	"	10-3	1.05			
"	12-3	9.6	"	12-3	1.9			
"	14-3	15.8	"	14-3	2.8			
"	16-3	20.3	"	16-3	3.2			
"	10-5	3.3	"	10-5	1.03			
"	12-5	7.6	"	12-5	2.1			

Summation of Pick Up Points

10-1 ———————— 59.52
12-1 ———————— 94.25
14-1 ————————118.91
16-1 ———————— 126.90
10-3 ———————— 56.82
12-3 ———————— 93.07
14-3 ———————— 112.57
16-3 ———————— 120.05
10-5 ———————— 48.06
12-5 ———————— 78.05
14-5 ———————— 95.08
16-5 ———————— 98.66
10-7 ———————— 31.05
12-7 ———————— 49.05
14-7 ———————— 58.90
16-7 ———————— 62.03

VOLTMETER READINGS FOR DETERMINING THE VOLUME OF THE VOLTAGE FUNCTION
FOR SQUARE SHAFT USING A 1 INCH INPUT GRID AND A 2 INCH PICK UP POINT GRID

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
A-I	10-1	2.8	A-J	14-7	2.5	A-L	10-7	1.0
"	12-1	6.6	"	16-7	2.8	"	12-7	1.7
"	14-1	13.0	A-K	10-1	3.4	"	14-7	1.8
"	16-1	20.0	"	12-1	8.8	"	16-7	1.9
"	10-3	2.4	"	14-1	13.0	A-M	10-1	5.2
"	12-3	5.9	"	16-1	10.6	"	12-1	10.8
"	14-3	10.2	"	10-3	3.1	"	14-1	6.6
"	16-3	13.0	"	12-3	6.6	"	16-1	5.7
"	10-5	1.7	"	14-3	8.8	"	10-3	2.7
"	12-5	3.8	"	16-3	8.2	"	12-3	6.0
"	14-5	6.1	"	10-5	2.0	"	14-3	5.5
"	16-5	6.7	"	12-5	3.8	"	16-3	5.0
"	10-7	0.90	"	14-5	5.1	"	10-5	1.9
"	12-7	1.80	"	16-5	5.2	"	12-5	3.2
"	14-7	2.8	"	10-7	0.90	"	14-5	3.4
"	16-7	3.0	"	12-7	1.7	"	16-5	3.4
A-J	10-1	3.0	"	14-7	2.25	"	10-7	0.90
"	12-1	7.2	"	16-7	2.50	"	12-7	1.4
"	14-1	14.1	A-L	10-1	4.5	"	14-7	1.6
"	16-1	14.8	"	12-1	7.9	"	16-7	1.5
"	10-3	2.7	"	14-1	9.5	A-N	10-1	7.8
"	12-3	6.2	"	16-1	7.8	"	12-1	6.0
"	14-3	10.2	"	10-3	3.4	"	14-1	4.15
"	16-3	11.5	"	12-3	7.0	"	16-1	3.3
"	10-5	1.9	"	14-3	7.40	"	10-3	4.1
"	12-5	4.0	"	16-3	6.4	"	12-3	4.6
"	14-5	5.7	"	10-5	2.0	"	14-3	3.65
"	16-5	6.3	"	12-5	3.6	"	16-3	2.3
"	10-7	1.0	"	14-5	4.45	"	10-5	1.6
"	12-7	1.8	"	16-5	4.3	"	12-5	1.1

Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.	Input Point	Pick up Point	Voltmeter A.C.
A-N	14-5	2.4	A-P	16-5	0.55	B-J	10-7	1.0
"	16-5	2.3	"	10-7	0.20	"	12-7	1.8
"	10-7	0.79	"	12-7	0.30	"	14-7	3.8
"	12-7	1.1	"	14-7	0.30	"	16-7	3.0
"	14-7	1.2	"	16-7	0.30	B-K	10-1	3.2
"	16-7	1.1	B-I	10-1	2.8	"	12-1	7.1
A-O	10-1	7.4	"	12-1	6.3	"	14-1	10.0
"	12-1	3.5	"	14-1	11.2	"	16-1	9.6
"	14-1	2.35	"	16-1	14.0	"	10-3	2.9
"	16-1	2.1	"	10-3	2.5	"	12-3	6.6
"	10-3	3.2	"	12-3	5.6	"	14-3	10.3
"	12-3	2.6	"	14-3	10.1	"	16-3	8.2
"	14-3	2.0	"	16-3	14.5	"	10-5	1.85
"	16-3	1.85	"	10-5	1.8	"	12-5	4.0
"	10-5	1.2	"	12-5	3.8	"	14-5	5.7
"	12-5	1.6	"	14-5	6.3	"	16-5	6.6
"	14-5	1.35	"	16-5	7.5	"	10-7	1.0
"	16-5	1.20	"	10-7	1.0	"	12-7	1.6
"	10-7	0.51	"	12-7	2.0	"	14-7	2.4
"	12-7	0.68	"	14-7	2.8	"	16-7	2.6
"	14-7	0.75	"	16-7	3.4	B-L	10-1	3.80
"	16-7	0.75	B-J	10-1	2.8	"	12-1	7.9
A-P	10-1	2.4	"	12-1	6.4	"	14-1	8.15
"	12-1	1.2	"	14-1	11.4	"	16-1	7.2
"	14-1	0.94	"	16-1	12.1	"	16-3	3.10
"	16-1	0.84	"	10-3	2.5	"	12-3	7.8
"	10-3	1.2	"	12-3	5.9	"	14-3	7.3
"	12-3	1.05	"	14-3	11.0	"	16-3	6.2
"	14-3	0.84	"	16-3	11.6	"	10-5	2.15
"	16-3	0.65	"	10-5	1.9	"	12-5	4.00
"	10-5	0.53	"	12-5	4.0	"	14-5	4.35
"	12-5	0.63	"	14-5	6.2	"	16-5	4.25
"	14-5	0.58	"	16-5	6.8	"	10-7	1.0

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
B-L	12-7	1.7	B-N	16-7	1.1	C-I	12-1	0.53
"	14-7	2.0	B-O	10-1	3.9	"	14-1	0.85
"	16-7	2.25	"	12-1	3.1	"	16-1	10.6
B-M	10-1	4.4	"	14-1	2.4	"	10-3	2.1
"	12-1	7.5	"	16-1	2.1	"	12-3	5.2
"	14-1	6.0	"	10-3	4.8	"	14-3	8.9
"	16-1	5.3	"	12-3	2.8	"	16-3	13.0
"	10-3	3.8	"	14-3	2.2	"	10-5	1.6
"	12-3	7.5	"	16-5	1.0	"	12-5	4.0
"	14-3	5.4	"	10-5	1.5	"	14-5	6.9
"	16-3	4.8	"	12-5	1.7	"	16-5	8.6
"	10-5	2.2	"	14-5	1.5	"	10-7	1.0
"	12-5	3.4	"	16-5	1.4	"	12-7	2.0
"	14-5	3.4	"	10-7	0.59	"	14-7	3.0
"	16-5	3.4	"	12-7	0.83	"	16-7	3.4
"	10-7	0.90	"	14-7	0.82	C-J	10-1	2.5
"	12-7	1.5	"	16-7	0.75	"	12-1	5.4
"	14-7	1.5	B-P	10-1	1.70	"	14-1	8.5
"	16-7	1.6	"	12-1	1.20	"	16-1	9.6
B-N	10-1	4.9	"	14-1	0.98	"	10-3	2.3
"	12-1	5.1	"	16-1	0.88	"	12-3	5.4
"	14-1	39	"	10-3	1.45	"	14-3	10.0
"	16-1	35	"	12-3	1.10	"	16-3	10.5
"	10-3	5.7	"	14-3	0.88	"	10-5	1.7
"	12-3	4.6	"	16-3	0.79	"	12-5	4.0
"	14-3	3.55	"	10-5	0.72	"	14-5	6.9
"	16-3	3.1	"	12-5	0.71	"	16-5	7.8
"	10-5	2.0	"	14-5	0.64	"	10-7	0.90
"	12-5	2.5	"	16-5	0.57	"	12-7	2.0
"	14-5	2.4	"	10-7	0.25	"	14-7	2.8
"	16-5	2.3	"	12-7	0.34	"	16-7	3.5
"	10-7	0.82	"	14-7	0.33	C-K	10-1	2.8
"	12-7	1.1	"	16-7	0.31	"	12-1	5.8
"	14-7	1.1	C-I	10-1	0.22	"	14-1	7.6

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
C-K	16-1	7.7	C-M	10-3	3.5	C-O	12-3	2.5
"	10-3	2.6	"	12-3	6.9	"	14-3	1.9
"	12-3	6.2	"	14-3	5.0	"	16-3	1.7
"	14-3	9.0	"	16-3	4.4	"	10-5	2.3
"	16-3	7.5	"	10-5	2.5	"	12-5	1.9
"	10-5	2.0	"	12-5	4.0	"	14-5	1.4
"	12-5	4.4	"	14-5	3.5	"	16-5	1.2
"	14-5	5.9	"	16-5	3.1	"	10-7	0.77
"	16-5	5.4	"	10-7	1.0	"	12-7	0.92
"	10-7	1.05	"	12-7	1.8	"	14-7	0.83
"	12-7	2.0	"	14-7	1.8	"	16-7	0.77
"	14-7	2.3	"	16-7	1.6	C-P	10-1	0.90
"	16-7	2.7	C-N	10-1	3.01	"	12-1	0.83
C-L	10-1	2.9	"	12-1	3.9	"	14-1	0.69
"	12-1	5.7	"	14-1	3.4	"	16-1	0.63
"	14-1	6.2	"	16-1	3.2	"	10-3	1.2
"	16-1	6.1	"	10-3	4.3	"	12-3	0.81
"	10-3	2.9	"	12-3	4.3	"	14-3	0.62
"	12-3	6.9	"	14-3	3.2	"	16-3	0.58
"	14-3	6.6	"	16-3	3.0	"	10-5	0.82
"	16-3	5.8	"	10-5	2.7	"	12-5	0.61
"	10-5	2.25	"	12-5	3.1	"	14-5	0.50
"	12-5	4.7	"	14-5	2.5	"	16-5	0.46
"	14-5	5.0	"	16-5	2.2	"	10-7	0.27
"	16-5	4.4	"	10-7	0.90	"	12-7	0.31
"	10-7	1.1	"	12-7	1.2	"	14-7	0,27
"	12-7	1.9	"	14-7	1.2	"	16-7	0.25
"	14-7	2.2	"	16-7	1.2	D-I	10-1	0.20
"	16-7	2.4	C-O	10-1	2.3	"	12-1	4.2
C-M	10-1	3.0	"	12-1	2.5	"	14-1	6.6
"	12-1	5.2	"	14-1	2.1	"	16-1	7.8
"	14-1	5.1	"	16-1	1.9	"	10-3	2.0
"	16-1	4.7	"	10-3	4.2	"	12-3	4.2

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
D-I	14-3	7.2	D-K	16-3	5.8	D-M	10-5	3.0
"	16-3	9.6	"	10-5	1.8	"	12-5	5.7
"	10-5	1.6	"	12-5	4.7	"	14-5	3.4
"	12-5	3.6	"	14-5	7.8	"	16-5	2.9
"	14-5	7.2	"	16-5	5.7	"	10-7	1.4
"	16-5	11.4	"	10-7	1.0	"	12-7	2.0
"	10-7	1.0	"	12-7	2.2	"	14-7	1.8
"	12-7	2.1	"	14-7	3.0	"	16-7	1.7
"	14-7	3.6	"	16-7	3.1	D-N	10-1	2.0
"	16-7	4.3	D-L	10-1	2.4	"	12-1	2.8
D-J	10-1	2.0	"	12-1	4.1	"	14-1	2.7
"	12-1	4.3	"	14-1	4.9	"	16-1	2.6
"	14-1	6.5	"	16-1	4.9	"	10-3	2.8
"	16-1	7.2	"	10-3	2.6	"	12.3	3.2
"	10-3	1.9	"	12-3	5.0	"	14-3	2.7
"	12-3	4.4	"	14-3	5.3	"	16-3	2.4
"	14-3	7.1	"	16-3	4.8	"	10-5	4.3
"	16-3	8.1	"	10-5	2.3	"	12-5	3.3
"	10-5	1.6	"	12-5	6.6	"	14-5	2.3
"	12-5	3.8	"	14-5	5.2	"	16-5	2.0
"	14-5	8.60	"	16-5	4.1	"	10-7	1.2
"	16-5	8.1	"	10-7	1.2	"	12-7	1.5
"	10-7	1.0	"	12-7	2.5	"	14-7	1.2
"	12-7	2.2	"	14-7	1.9	"	16-7	1.1
"	14-7	3.50	"	16-7	2.4	D-O	10-1	1.4
"	16-7	4.2	D-M	10-1	2.1	"	12-1	1.8
D-K	10-1	2.2	"	12-1	3.5	"	14-1	1.7
"	12-1	4.2	"	14-1	3.7	"	16-1	1.6
"	14-1	5.8	"	16-1	3.5	"	10-3	2.3
"	16-1	6.0	"	10-3	2.6	"	12-3	2.0
"	10-3	2.2	"	12-3	4.0	"	14-3	1.65
"	12-3	4.6	"	14-3	3.8	"	16-3	1.5
"	14-3	6.4	"	16-3	3.5	"	10-5	3.8

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
D-O	12-5	2.0	E-I	16-5	10.5	E-K	10-7	1.0
"	14-5	1.4	"	10-7	0.90	"	12-7	2.6
"	16-5	1.2	"	12-7	2.0	"	14-7	3.6
"	10-7	1.10	"	14-7	4.0	"	16-7	3.0
"	12-7	1.0	"	16-7	6.0	E-L	10-1	1.6
"	14-7	0.85	E-J	10-1	1.6	"	12-1	2.85
"	16-7	0.75	"	12-1	3.2	"	14-1	3.6
D-P	10-1	0.76	"	14-1	4.7	"	16-1	3.6
"	12-1	0.85	"	16-1	5.0	"	10-3	1.7
"	14-1	0.78	"	10-3	1.5	"	12-3	3.05
"	16-1	0.73	"	12-3	3.4	"	14-3	3.8
"	10-3	1.1	"	14-3	5.1	"	16-3	3.6
"	12-3	0.90	"	16-3	5.4	"	10-5	2.0
"	14-3	0.75	"	10-5	1.4	"	12-5	5.6
"	16-3	0.67	"	12-5	3.3	"	14-5	4.4
"	10-5	1.30	"	14-5	7.1	"	16-5	3.4
"	12-5	0.78	"	16-5	7.1	"	10-7	1.3
"	14-5	0.61	"	10-7	1.0	"	12-7	3.0
"	16-5	0.55	"	12-7	2.2	"	14-7	2.7
"	10-7	0.52	"	14-7	4.4	"	16-7	2.3
"	12-7	0.45	"	16-7	4.4	E-M	10-1	1.4
"	14-7	0.35	E-K	10-1	1.5	"	12-1	2.4
"	16-7	0.31	"	12-1	3.2	"	14-1	2.7
E-I	10-1	1.6	"	14-1	4.1	"	16-1	2.8
"	12-1	3.1	"	16-1	4.4	"	10-3	1.7
"	14-1	4.8	"	10-3	1.7	"	12-3	2.9
"	16-1	5.3	"	12-3	3.4	"	14-3	2.85
"	10-3	1.6	"	14-3	4.6	"	16-3	2.8
"	12-3	3.3	"	16-3	4.6	"	10-5	2.7
"	14-3	5.2	"	10-5	1.6	"	12-5	4.9
"	16-3	6.2	"	12-5	4.2	"	14-5	2.9
"	10-5	1.2	"	14-5	6.6	"	16-7	2.4
"	12-5	3.2	"	16-5	5.0	"	10-7	1.7
"	14-5	6.0				"	12-7	2.6

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
E-M	14-7	1.8	E-P	10-1	0.53	F-J	14-1	2.9
"	16-7	1.4	"	12-1	0.66	"	16-1	3.2
E-N	10-4	1.3	"	14-1	0.65	"	10-3	1.1
"	12-1	1.9	"	16-1	0.63	"	12-3	2.2
"	14-1	2.1	"	10-3	0.75	"	14-3	3.2
"	16-1	2.1	"	12-3	0.73	"	16-3	3.4
"	10-3	1.65	"	14-3	0.65	"	10-5	1.1
"	12-3	2.3	"	16-3	0.61	"	12-3	2.3
"	14-3	2.2	"	10-5	1.35	"	14-5	4.2
"	16-3	2.05	"	12-5	0.81	"	16-3	4.8
"	10-5	3.5	"	14-5	0.58	"	10-7	0.82
"	12-5	3.2	"	16-5	0.52	"	12-7	1.8
"	14-5	2.05	"	10-7	0.82	"	14-7	5.8
"	16-5	1.90	"	12-7	0.54	"	16-7	5.2
"	10-7	2.0	"	14-7	0.37	F-K	10-1	1.0
"	12-7	2.0	"	16-7	0.31	"	12-1	2.0
"	14-7	1.2	F-I	10-1	1.2	"	14-1	2.8
"	16-7	1.2	"	12-1	2.0	"	16-1	3.2
E-O	10-1	1.0	"	14-1	3.1	"	10-3	1.1
"	12-1	1.2	"	16-1	3.5	"	12-3	2.3
"	14-1	1.2	"	10-3	1.1	"	14-3	3.1
"	16-1	1.2	"	12-3	2.4	"	16-3	3.3
"	10-3	1.2	"	14-3	3.6	"	10-5	1.2
"	12-3	1.3	"	16-3	3.7	"	12-5	2.9
"	14-3	1.2	"	10-5	1.1	"	14-5	4.0
"	16-3	1.1	"	12-5	2.3	"	16-3	3.9
"	10-5	3.4	"	14-5	4.3	"	10-7	0.90
"	12-5	1.6	"	16-5	6.4	"	12-7	2.6
"	14-5	1.15	"	10-7	0.70	"	14-7	4.6
"	16-5	1.08	"	12-7	1.7	"	16-7	3.0
"	10-7	1.65	"	14-7	4.25	F-L	10-1	1.0
"	12-7	1.0	"	16-7	10.5	"	12-1	2.0
"	14-7	0.76	F-J	10-1	1.18	"	14-1	2.4
"	16-7	0.65	"	12-1	2.0	"	16-1	2.5

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
F-L	10-3	1.2	F-N	12-3	1.4	F-P	14-3	0.34
"	12-3	2.0	"	14-3	1.5	"	16-3	0.34
"	14-3	2.6	"	16-3	1.45	"	10-5	0.68
"	16-3	2.6	"	10-5	2.05	"	12-5	0.46
"	10-5	1.4	"	12-5	2.2	"	14-5	0.32
"	12-5	3.1	"	14-5	1.6	"	16-5	0.30
"	14-5	3.2	"	16-5	1.3	"	10-7	0.92
"	16-5	3.7	"	10-7	3.2	"	12-7	0.39
"	10-7	1.2	"	12-7	2.4	"	14-7	0.24
"	12-7	4.2	"	14-7	1.1	"	16-7	0.20
"	14-7	2.8	"	16-7	0.95	G-I	10-1	0.73
"	16-7	2.1	F-O	10-1	0.67	"	12-1	1.2
F-M	10-1	1.0	"	12-1	1.08	"	14-1	1.8
"	12-1	1.6	"	14-1	1.0	"	16-1	2.0
"	14-1	1.8	"	16-1	0.97	"	10-3	0.73
"	16-1	1.9	"	10-3	0.84	"	12-3	1.25
"	10-3	1.2	"	12-3	1.08	"	14-3	1.95
"	12-3	1.8	"	14-3	1.0	"	16-3	2.15
"	14-3	2.0	"	16-3	0.94	"	10-5	0.68
"	16-3	1.9	"	10-5	1.8	"	12-5	1.4
"	10-5	1.8	"	12-5	1.2	"	14-5	0.24
"	12-5	2.8	"	14-5	0.95	"	16-5	3.1
"	14-5	2.2	"	16-5	0.84	"	10-7	0.50
"	16-5	1.9	"	10-7	3.2	"	12-7	1.1
"	10-7	2.1	"	12-7	1.6	"	14-7	3.1
"	12-7	3.8	"	14-7	0.68	"	16-7	5.8
"	14-7	1.6	"	16-7	0.53	G-J	10-1	0.71
"	16-7	1.2	F-P	10-1	0.25	"	12-1	1.2
F-N	10-1	9.2	"	12-1	0.33	"	14-1	1.6
"	12-1	1.3	"	14-1	0.35	"	16-1	1.8
"	14-1	1.4	"	16-1	0.35	"	10-3	0.70
"	16-1	1.45	"	10-3	0.32	"	12-3	1.3
"	10-3	1.00	"	12-3	0.37	"	14.3	1.8

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
G-J	16-3	2.0	G-L	10-5	0.98	G-N	12-5	1.2
"	10-5	0.74	"	12-5	1.7	"	14-5	1.05
"	12-5	1.4	"	14-5	1.6	"	16-5	0.91
"	14-5	2.4	"	16-5	1.6	"	10-7	2.65
"	16-5	2.8	"	10-7	1.05	"	12-7	1.8
"	10-7	0.54	"	12-7	3.2	"	14-7	0.86
"	12-7	1.3	"	14-7	2.2	"	16-7	0.63
"	14-7	3.8	"	16-7	1.4	G-O	10-1	0.35
"	16-7	3.8	G-M	10-1	6.0	"	12-1	0.53
G-K	10-1	0.73	"	12-1	1.0	"	14-1	0.57
"	12-1	1.0	"	14-1	1.0	"	16-1	0.58
"	14-1	1.5	"	16-1	1.1	"	10-3	0.45
"	16-1	1.6	"	10-3	0.72	"	12-3	0.58
"	10-3	0.72	"	12-3	1.13	"	14-3	0.60
"	12-3	1.2	"	14-3	1.2	"	16-3	0.57
"	14-3	1.8	"	16-3	1.1	"	10-5	0.85
"	16-3	1.7	"	10-5	1.05	"	12-5	0.80
"	10-5	0.80	"	12-5	1.5	"	14-5	0.60
"	12-5	1.5	"	14-5	1.25	"	16-5	0.53
"	14-5	2.05	"	16-5	1.2	"	10-7	2.75
"	16-5	2.2	"	10-7	1.6	"	12-7	0.96
"	10-7	0.71	"	12-7	3.0	"	14-7	0.45
"	12-7	1.9	"	14-7	1.2	"	16-7	0.34
"	14-7	3.6	"	16-7	0.94	G-P	10-1	0.15
"	16-7	2.2	G-N	10-1	0.56	"	12-1	0.21
G-L	10-1	0.70	"	12-1	0.85	"	14-1	0.22
"	12-1	1.2	"	14-1	0.95	"	16-1	0.22
"	14-1	1.3	"	16-1	0.94	"	10-3	0.18
"	16-1	1.4	"	10-3	0.62	"	12-3	0.23
"	10-3	0.77	"	12-3	0.96	"	14-3	0.22
"	12-3	1.3	"	14-3	1.00	"	16-3	0.21
"	14-3	1.5	"	16-3	0.95	"	10-5	0.36
"	16-3	1.55	"	10-5	1.1	"	12-5	0.30

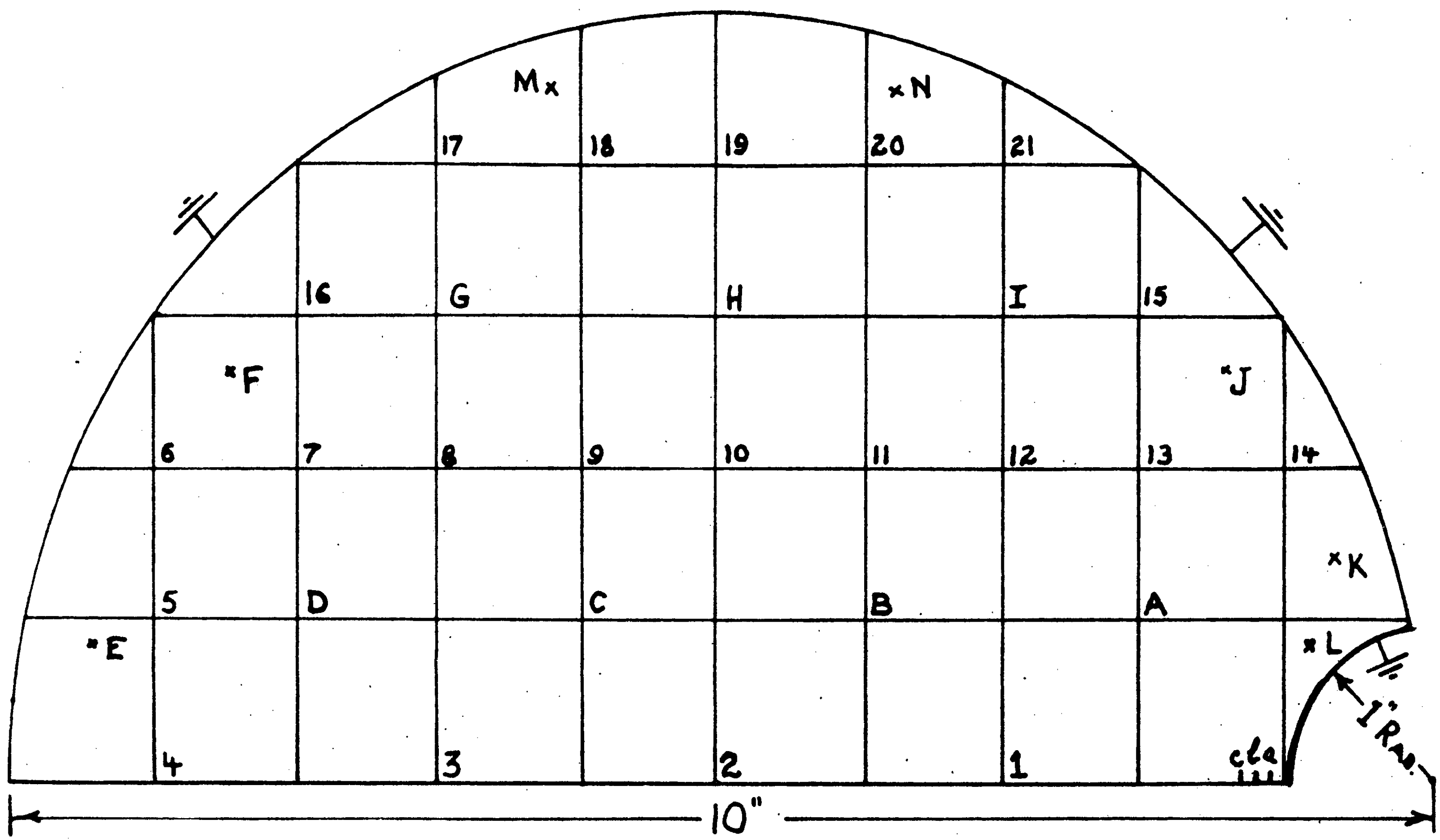
Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
G-P	14-5	0.22	H-J	16-5	0.88	H-L	10-7	0.34
"	16-5	0.20	"	10-7	0.20	"	12-7	0.87
"	10-7	0.85	"	12-7	0.50	"	14-7	0.71
"	12-7	0.31	"	14-7	1.1	"	16-7	0.48
"	14-7	0.16	"	16-7	1.2	H-M	10-1	0.21
"	16-7	0.12	H-K	10-1	0.24	"	12-1	0.35
H-I	10-1	0.22	"	12-1	0.42	"	14-1	0.40
"	12-1	0.41	"	14-1	0.54	"	16-1	0.42
"	14-1	0.56	"	16-1	0.57	"	10-3	0.25
"	16-1	0.62	"	10-3	0.25	"	12-3	0.38
"	10-3	0.21	"	12-3	0.46	"	14-3	0.45
"	12-3	0.44	"	14-3	0.58	"	16-3	0.45
"	14-3	0.62	"	16-3	0.61	"	10-5	0.37
"	16-3	0.69	"	10-5	0.27	"	12-5	0.56
"	10-5	0.71	"	12-5	0.55	"	14-5	0.51
"	12-5	0.47	"	14-5	0.72	"	16-5	0.46
"	14-5	0.76	"	16-5	0.75	"	10-7	0.66
"	16-5	0.93	"	10-1	0.26	"	12-7	1.05
"	10-7	0.16	"	12-7	0.72	"	14-7	0.50
"	12-7	0.40	"	14-7	1.1	"	16-7	0.36
"	14-7	1.0	"	16-7	0.79	H-N	10-1	0.19
"	16-7	1.6	H-L	10-1	0.21	"	12-1	0.30
H-J	10-1	0.23	"	12-1	0.36	"	14-1	0.32
"	12-1	0.42	"	14-1	0.44	"	16-1	0.34
"	14-1	0.57	"	16-1	0.47	"	10-3	0.21
"	16-1	0.61	"	10-3	0.22	"	12-3	0.32
"	10-3	0.24	"	12-3	0.39	"	14-3	0.35
"	12-3	0.45	"	14-3	0.48	"	16-3	0.33
"	14-3	0.62	"	16-3	0.50	"	10-5	0.38
"	16-3	0.65	"	10-5	0.30	"	12-5	0.46
"	10-5	0.24	"	12-5	0.52	"	14-5	0.37
"	12-5	0.50	"	14-5	0.58	"	16-5	0.32
"	14-5	0.79	"	16-5	0.54	"	10-7	0.90

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
H-N	12-7	0.74	H-P	14-7	.07
"	14-7	0.33	"	16-7	.05
"	16-7	0.24			
H-O	10-1	0.13			
"	12-1	0.20			
"	14-1	0.21			
"	16-1	0.21			
"	10-3	0.15			
"	12-3	0.22			
"	14-3	0.23			
"	16-3	0.205			
"	10-5	0.31			
"	12-5	0.30			
"	14-5	0.22			
"	16-5	0.20			
"	10-7	0.85			
"	12-7	0.38			
"	14-7	0.17			
"	16-7	0.12			
H-P	10-1	0.05			
"	12-1	0.09			
"	14-1	0.09			
"	16-1	0.09			
"	10-3	0.07			
"	12-3	0.10			
"	14-3	0.09			
"	16-3	0.08			
"	10-5	0.15			
"	12-5	0.14			
"	14-5	0.10			
"	16-5	0.08			
"	10-7	0.39			
"	12-7	0.15			

SUMMATION OF PICK UP POINTS

10-1	122.30
12-1	197.41
14-1	244.11
16-1	253.47
10-3	116.30
12-3	193.00
14-3	235.87
16-3	242.08
10-5	99.57
12-5	161.63
14-5	195.85
16-5	212.52
10-7	65.99
12-7	99.74
14-7	119.24
16-7	125.55

GRID SYSTEM FOR CIRCULAR SHAFT WITH KEYWAY
IN TORSION



NOTE: a, b, c MEASURED AT $\frac{1}{10}$" INCREMENTS FROM GROUNDED EDGE

VOLTMETER READINGS FOR DETERMINING THE VOLUME OF THE VOLTAGE FUNCTIONAL FOR CIRCULAR SHAFT WITH CIRCULAR KEYWAY WITH A 2-INCH INPUT GRID AND A 2-INCH PICKUP GRID

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
A	1	8.7	"	10	7.1	"	19	2.0
"	2	3.9	"	11	8.2	"	20	1.5
"	3	1.6	"	12	6.1	"	21	0.85
"	4	0.5	"	13	3.2	D	1	1.5
"	5	0.45	"	14	1.2	"	2	3.8
"	6	0.27	"	15	1.6	"	3	8.6
"	7	0.75	"	16	1.0	"	4	6.5
"	8	1.2	"	17	0.8	"	5	6.0
"	9	1.9	"	18	1.3	"	6	2.8
"	10	2.8	"	19	1.7	"	7	6.1
"	11	3.9	"	20	1.6	"	8	5.6
"	12	5.4	"	21	1.2	"	9	4.2
"	13	5.6	C	1	4.3	"	10	2.9
"	14	2.2	"	2	11.0	"	11	2.0
"	15	1.9	"	3	11.4	"	12	1.3
"	16	0.43	"	4	2.9	"	13	0.70
"	17	0.35	"	5	2.5	"	14	0.25
"	18	0.62	"	6	1.4	"	15	0.42
"	19	0.92	"	7	4.0	"	16	2.2
"	20	1.40	"	8	7.2	"	17	0.90
"	21	0.85	"	9	9.4	"	18	1.1
B	1	8.7	"	10	7.7	"	19	0.95
"	2	10.5	"	11	5.4	"	20	0.69
"	3	4.4	"	12	3.3	"	21	0.35
"	4	1.3	"	13	1.7	E	1	0.16
"	5	1.2	"	14	0.63	"	2	0.38
"	6	0.63	"	15	1.02	"	3	0.84
"	7	1.7	"	16	2.1	"	4	1.7
"	8	3.1	"	17	1.3	"	5	2.0
"	9	4.9	"	18	1.9	"	6	0.60

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
E	7	0.69	F	19	0.19	H	10	7.3
"	8	0.55	"	20	0.11	"	11	5.4
"	9	0.41	"	21	0.05	"	12	3.2
"	10	0.30	G	1	1.4	"	14	1.6
"	11	0.21	"	2	2.9	"	14	0.57
"	12	0.14	"	3	3.6	"	15	1.15
"	13	0.05	"	4	1.4	"	16	1.2
"	14	0	"	5	1.4	"	17	1.3
"	15	0	"	6	1.2	"	18	3.3
"	16	0.26	"	7	3.8	"	19	4.8
"	17	0.1	"	8	6.2	"	20	3.1
"	18	0.11	"	9	4.9	"	21	1.3
"	19	0.10	"	10	3.2	I	1	3.1
"	20	0.05	"	11	2.1	"	2	2.9
"	21	0	"	12	1.3	"	3	1.6
F	1	0.23	"	13	0.80	"	4	0.50
"	2	0.51	"	14	0.25	"	5	0.45
"	3	0.80	"	15	0.44	"	6	0.28
"	4	0.55	"	16	3.8	"	7	0.77
"	5	0.65	"	17	2.8	"	8	1.3
"	6	1.1	"	18	2.5	"	9	2.1
"	7	1.5	"	19	1.6	"	10	2.2
"	8	0.97	"	20	0.94	"	11	4.9
"	9	0.66	"	21	0.45	"	12	5.5
"	10	0.45	H	1	2.9	"	13	3.2
"	11	0.31	"	2	4.7	"	14	1.1
"	12	0.20	"	3	3.3	"	15	3.3
"	13	0.19	"	4	1.05	"	16	0.46
"	14	0.04	"	5	0.98	"	17	0.48
"	15	0.05	"	6	0.62	"	18	1.02
"	16	1.5	"	7	1.8	"	19	1.7
"	17	0.28	"	8	3.4	"	20	2.8
"	18	0.25	"	9	5.7	"	21	3.2

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
J	1	0.75	K	13	0.26	"	4	0.2
"	2	0.50	"	14	0.50	"	5	0.16
"	3	0.25	"	15	0.10	"	6	0.12
"	4	0.05	"	16	0	"	7	0.35
"	5	0.05	"	17	0	"	8	0.60
"	6	0	"	18	0	"	9	0.77
"	7	0.10	"	19	0	"	10	0.70
"	8	0.20	"	20	0.01	"	11	0.54
"	9	0.30	"	21	0.01	"	12	0.34
"	10	0.45	L	1	0.10	"	13	0.18
"	11	0.65	"	2	0.05	"	14	0.05
"	12	1.0	"	3	0	"	15	0.12
"	13	1.6	"	4	0	"	16	0.33
"	14	1.3	"	5	0	"	17	0.80
"	15	1.5	"	6	0	"	18	2.8
"	16	0.06	"	7	0	"	19	1.2
"	17	0.05	"	8	0	"	20	0.46
"	18	0.12	"	9	0	"	21	0.18
"	19	0.19	"	10	0.02	N	1	0.38
"	20	0.25	"	11	0.05	"	2	0.47
"	21	0.29	"	12	0.07	"	3	0.31
K	1	0.20	"	13	0.10	"	4	0.10
"	2	0.10	"	14	0.10	"	5	0.09
"	3	0.04	"	15	0.02	"	6	0.04
"	4	0	"	16	0	"	7	0.17
"	5	0	"	17	0	"	8	0.30
"	6	0	"	18	0	"	9	0.45
"	7	0	"	19	0	"	10	0.61
"	8	0	"	20	0	"	11	0.67
"	9	0.03	"	21	0	"	12	0.53
"	10	0.06	M	1	0.31	"	13	0.29
"	11	0.11	"	2	0.54	"	14	0.10
"	12	0.60	"	3	0.48	"	15	0.27

Input Point	Pick Up Point	Voltmeter A.C.
N	16	0.11
"	17	0.14
"	18	0.36
"	19	0.93
"	20	2.4
"	21	0.90

Summation of Pick Up Points

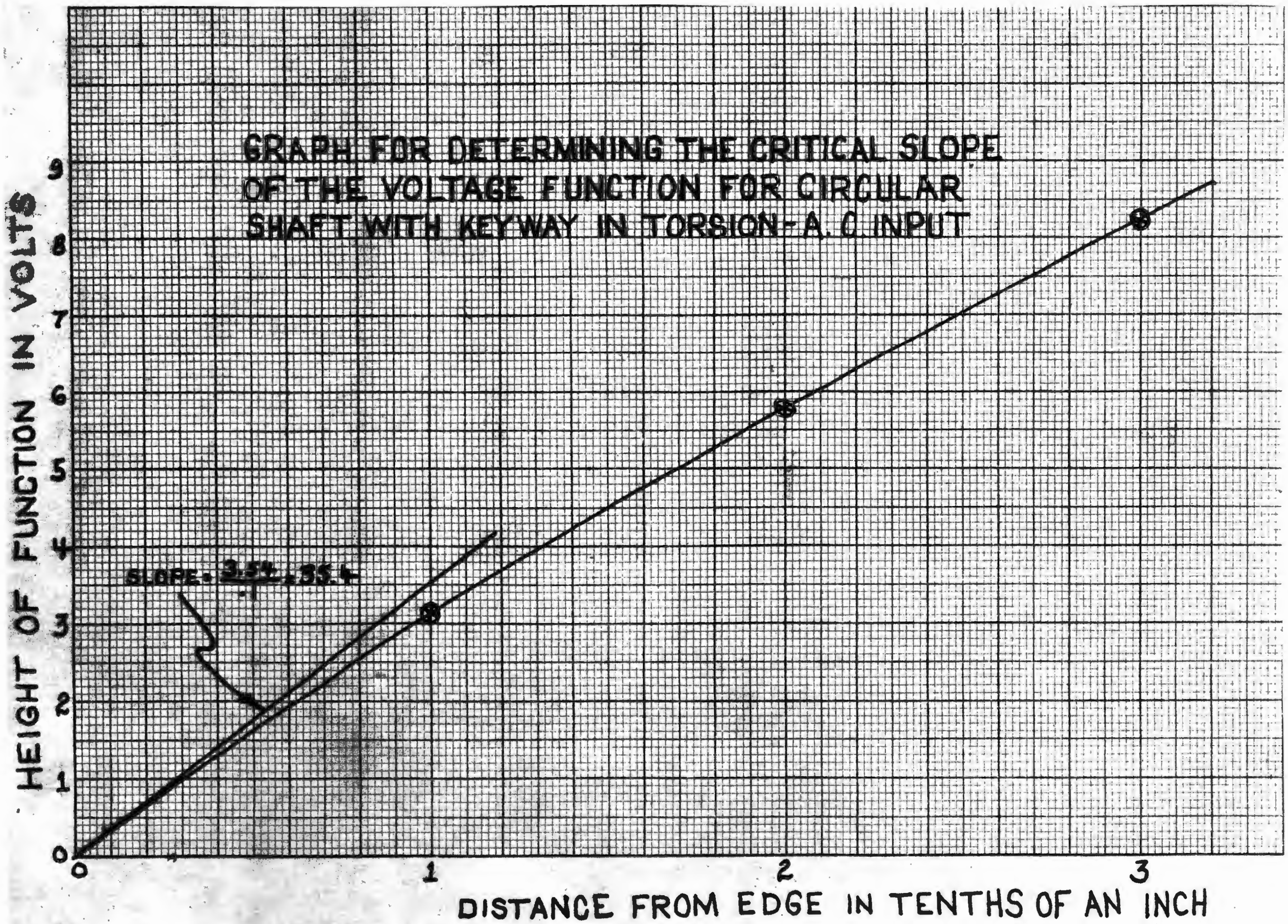
1 --- 32.73	8 --- 30.62	15 --- 11.89
2 --- 42.25	9 --- 35.72	16 --- 13.45
3 --- 37.22	10 --- 35.79	17 --- 9.30
4 --- 16.75	11 --- 34.53	18 --- 15.38
5 --- 15.93	12 ----28.98	19 --- 16.28
6 --- 9.06	13 --- 19.47	20 --- 14.90
7 --- 21.73	14 --- 8.30	21 --- 9.63

VOLTMETER READINGS FOR DETERMINING MAXIMUM SLOPE FOR A CIRCULAR SHAFT WITH A CIRCULAR KEYWAY IN TORSION WITH A 2-INCH INPUT GRID

Input Point	Pick Up Point	Voltmeter A.C.	Input Point	Pick Up Point	Voltmeter A.C.
A	a	1.4	H	a	0.22
"	b	2.5	"	b	0.41
"	c	3.5	"	c	0.59
B	a	0.61	I	a	0.29
"	b	1.18	"	b	0.56
"	c	1.60	"	c	0.82
C	a	0.29	J	a	0.08
"	b	0.55	"	b	0.14
"	c	0.81	"	c	0.21
D	a	0.08	K	a	0.021
"	b	0.16	"	b	0.040
"	c	0.24	"	c	0.062
E	a	0	L	a	0.020
"	b	0	"	b	0.035
"	c	0	"	c	0.050
F	a	0.01	M	a	0.007
"	b	0.02	"	b	0.014
"	c	0.03	"	c	0.020
G	a	0.08	N	a	0.012
"	b	0.15	"	b	0.020
"	c	0.235	"	c	0.035

Summation of Points

a ═══ 3.12
b ═══ 5.78
c ═══ 8.21



GRAPH FOR DETERMINING THE CRITICAL SLOPE
OF THE VOLTAGE FUNCTION FOR CIRCULAR
SHAFT WITH KEYWAY IN TORSION - A. C. INPUT
SLOPE = 3.54/.1 = 35.4
HEIGHT OF FUNCTION IN VOLTS
0
1
2
3
4
5
6
7
8
9
DISTANCE FROM EDGE IN TENTHS OF AN INCH
0
1
2
3

APPENDIX C

EXPERIMENTAL SOLUTIONS

D.C. VOLTMETER SOLUTION FOR MAXIMUM SHEARING STRESS FOR SQUARE BAR IN TORSION

Critical Slope (From graph in App. B) in volts/in. = $\frac{\partial v}{\partial N}$ = 14.8

Measured Resistance of paper in ohms/sq. = 1975

Current Density in amps/sq. in. = .00102

Scale Factor = 1

Substituting these values into Equation 10

$$T_{max.} = \frac{2G\theta}{mip} \frac{\partial v}{\partial N} = \frac{2G\theta\ (14.8)}{.00102\ (1975)} = 14.7\ G\theta$$

PERCENT ERROR

Mathematical answer (see App. A) = 10.8 $G\theta$

$$\%\ \text{Error} = \frac{14.7 - 10.8}{10.8} \times 100 = 36\%\ \text{high}$$

A.C. VOLTMETER SOLUTION FOR MAXIMUM SHEARING STRESS FOR SQUARE BAR IN TORSION

Critical Slope (From graph in App. B) in volts/in. = $\frac{\partial v}{\partial N}$ = 106.5

Measured resistance of paper in ohms/sq. = 1950

Current density in amps/sq. in. = .010

Scale factor = 1

Substituting these values into Equation 10

$$T_{max.} = \frac{2G\theta}{mip} \frac{\partial v}{\partial N} = \frac{2G\theta\ (106.5)}{.010\ (1950)} = 10.8\ G\theta$$

PERCENT ERROR

Mathematical solution (see App. A) = 10.8 Gθ

$$\%\ \text{Error} = \frac{10.8 - 10.8}{10.8} \times 100 = 0\%$$

A.C. VOLTMETER SOLUTION FOR TWISTING MOMENT FOR SQUARE BAR IN TORSION WITH 2 INCH INPUT AND 2 INCH PICKUP GRIDS.

Calculation of the volume of the voltage function

$$\text{Vol.} = \frac{\sum \text{Ordinates}}{4} \times \text{Area of grid square}$$

$\sum$ Ordinates = 126.90 + 2 (120.05 + 98.66 + 62.03 + 118.91 + 94.25 + 59.52) + 4 (58.90 + 95.08 + 112.57 + 93.07 + 78.05 + 49.05 + 31.05 + 48.06 + 56.82) = 3,724.34.

Volume for 1 quadrant = $\frac{(3.724.34)}{4}$ = 3724.34 volt-in.^{2}

Total volume = (3,724.34) (4) = 14.897.36 volt-in.^{2}

TWISTING MOMENT

Volume in volt-in.^{2}	= 14,897.36
Measured resistance of paper in ohms/sq.	= 1,450
Current density in amps/sq. in.	= .0050
Scale Factor	= 1

Substituting these values into equation 11-A

$$M_t = \frac{4G\theta}{m^2 ip} \text{ (vol. of V function)} = \frac{4G\theta \ (14,897.36)}{(.0050)\ (1450)} = 8,230\ G\theta$$

PERCENT ERROR

Mathematical answer (see App. A) = 9,240 $G\theta$

% Error $\frac{9240 - 8230}{9240} \times 100$ = 11% low

A.C. VOLTMETER SOLUTION FOR TWISTING MOMENT FOR SQUARE BAR IN TORSION WITH 1 INCH INPUT AND 2 INCH PICKUP GRIDS.

Calculation of the volume of the voltage function

$$\text{Vol.} = \frac{\sum \text{Ordinates}}{4} \times \text{area of grid square}$$

$\sum$ Ordinates = 253.47 + 2(242.08 + 218.22 + 125.55 + 244.11 + 197.41 + 122.30) + 4(119.24 + 195.85 + 235.87 + 193.00 + 161.63 + 99.74 + 65.99 + 99.594 + 116.30) = 7,701.65

$$\text{Volume for 1 quadrant} = \frac{(7701.65)\ (4)}{4} = 7701.65 \text{ volt-in.}^2$$

$$\text{Total volume} = (7{,}701.65)\ (4) = 30{,}806.6 \text{ volt-in.}^2$$

TWISTING MOMENT

Volume in volt-in.^{2}	= 30,806.6
Measured resistance of paper in ohms/sq.	= 1,450
Current density in amps/sq. in.	= .010
Scale Factor	= 1

Substituting these values into equation 11-A

$$M_t = \frac{4G\theta}{m^2ip} \text{ (vol. of V function)} = \frac{4G\theta\ (30806.6)}{(.010)\ (1450)} = 8{,}500\ G\theta$$

PERCENT ERROR

Mathematical answer (see App. A) = 9240 $G\theta$

$$\%\ \text{Error} = \frac{9240 - 8500}{9240} \times 100 = 8\%\ \text{low}$$

A.C. VOLTMETER SOLUTION FOR MAXIMUM SHEARING STRESS FOR CIRCULAR SHAFT WITH KEYWAY IN TORSION WITH 2 INCH INPUT AND 2 INCH PICKUP GRID

Critical slope (from graph in App. B) = $\frac{\partial v}{\partial N}$ = 33.0

Measured resistance of paper in ohms/sq. = 1610

Current density in amps/sq. in. = .0050

Scale Factor = 1

Substituting these values into equation 10

$$T_{max.} = \frac{2G\theta}{mip} \frac{\partial v}{\partial N} = \frac{2G\theta\ (35.4)}{.005\ (1610)} = 8.80\ G\theta$$

PERCENT ERROR

Mathematical answer (see App. A) = 9.00 Gθ

$$\%\ \text{Error} = \frac{9.00 - 8.80}{9.00} \times 100 = 2.2\%\ \text{low}$$

A.C. VOLTMETER SOLUTION FOR TWISTING MOMENT FOR CIRCULAR SHAFT WITH KEYWAY IN TORSION WITH 2 INCH INPUT AND 2 INCH PICKUP GRIDS.

Calculation of the volume of the voltage function

$$v = \frac{\sum \text{Ordinates}}{4} \times \text{area of grid square}$$

$$\sum_1 \text{Ordinates} = 16.75 + 0.96 + 8.30 + 19.47 + 21.73 + 2(37.22 + 30.62 + 42.25 + 32.73 + 28.98 + 34.53 + 14.90 + 15.38 + 35.72) = 691.55$$

$$v_1 = \frac{691.55}{4}(4) = 691.55 \text{ volt-in.}^2$$

The volumes of the partial edge squares must now be added.

$$\sum_2 = 21.73 + 9.06 + 13.45 + 8.30 + 19.47 + 11.89 = 83.90$$

$$v_2 = \frac{83.90}{4}(1) = 20.95 \text{ volt-in.}^2$$

$$\sum_3 = 15.93 + 9.06 + 8.30 + 5.00 = 38.23$$

$$v_3 = \frac{38.23}{4}(.75) = 7.17 \text{ volt-in.}^2$$

$$v_4 = \frac{16.75 + 15.93}{4}(.95) = 7.53 \text{ volt-in.}^2$$

$$v_5 = \frac{14.90 + 16.28 + 16.28 + 15.38}{4}(.95) = 14.93 \text{ volt-in.}^2$$

$$v_6 = \frac{15.38 + 9.30 + 14.90 + 9.63}{4}(.72) = 8.85 \text{ volt-in.}^2$$

The following volumes we figured as segments of a pyramid, where volume = 1/3 (Area Base)h (NOTE: Average heights were used for symmetrical sections)

$$v_7 = \frac{8.30 + 9.06}{2} \times 1/3\,(.58) = 3.44 \text{ volt-in.}^2$$

$$v_8 = \frac{13.45 + 11.89}{2} \times 1/3\,(1.0) = 8.40 \text{ volt-in.}^2$$

$$v_9 = \frac{9.30 + 9.63}{2} \times 1/3\ (.42) = 5.30 \text{ volt-in.}^2$$

$$\text{volume of 1/2 the voltage function} = \sum_{i=1}^{8} v_i = 768.12 \text{ volt-in.}^2$$

$$\text{by symmetry the total volume} = (768.12)\ (2) = 1536.24 \text{ volt-in.}^2$$

TWISTING MOMENT

Volume in volt-in.^{2}	= 1536.24
Measured resistance of the paper in ohms/sq.	= 1610
Current density in amps/sq. in.	= .0050
Scale Factor	= 1

Substituting into equation 11-A

$$M_t = \frac{4G\theta}{m^2 i p}\ (\text{vol. of v function}) = \frac{4G\theta\ (1536.24)}{(.0050)\ (1610)} = 763\ G\theta$$

PERCENT ERROR

Mathematical answer (see App. A) = 800 $G\theta$

$$\%\ \text{Error} = \frac{800 - 763}{800} = 4.6\%\ \text{low}$$

RATIO OF M_t TO $T_{max.}$

$$\text{Mathematical} = \frac{800\ G\theta}{9.00\ G\theta} = 88.9$$

$$\text{Experimental} = \frac{763\ G\theta}{8.80\ G\theta} = 86.7$$

$$\%\ \text{Error} = \frac{88.9 - 86.7}{88.9} \times 100 = 2.5\%\ \text{low}$$

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VITA

Ronald Edward Volker was born on June 27, 1933 at Buffalo, New York, the son of William J. and Mary L. Volker. He attended public schools in Lancaster, New York, graduating from high school in 1951.

For one year after high school he worked as an apprentice draftsman for the Symington Gould Corp. and later as a senior technical clerk with the American Machine and Foundry Corp. In 1953 he was drafted into the United States Army and served with the Transportation Corps as a chief construction surveyor.

Upon his release from the United States Army, he entered the Missouri School of Mines and Metallurgy and graduated with a Bachelor of Science Degree in Civil Engineering in August, 1959. He then received a National Science Foundation Fellowship to study for a Master of Science Degree in Civil Engineering.

On August 4, 1956 he was united in marriage to Patricia Ann Frost of Lancaster, New York. A son, Mark Andrew, was born on May 22, 1957 and a second son, James Allan, on June 20, 1958. Twin sons, Peter Carl and Paul Richard, were born on December 18, 1959.